Math 5615 Honors: Series

Sasha Voronov

University of Minnesota

October 21, 2020

The Sequence $\{\sqrt[n]{n}\}$

Theorem

The sequence $\{n^{1/n}\}$ *for* $n \geq 4$ *is decreasing and convergent, and* $\lim_{n\to\infty} n^{1/n} = 1$*.*

Proof. Step 1: Sequence is ~~in~~decreasing, starting with $n \geq 4$.

$$\left(1+\frac{1}{n}\right)^n < 4 \quad \text{(from last time)} \quad \forall n \geq 1$$

$$\left(\frac{n+1}{n}\right)^n < n \quad \forall n \geq 4$$

$$(n+1)^n < n^{n+1}$$

$$(n+1) < n^{\frac{n+1}{n}}$$

$$(n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}}$$

Continuing with Proof of the Limit $\{\sqrt[n]{n}\}$

Step 2: Sequence is bounded below by 1. Therefore, it has a limit $L = \overset{\text{inf}}{\cancel{\sup}}\{n^{1/n} \mid n \geq 4\} \geq 1$.

Step 3: $L = 1$. Idea: show the sequence $\{n^{1/n} = (2n/2)^{2/2n}\}$ converges to L^2. We have $\lim_{n\to\infty} n^{2/n} = L^2$. Then

$$\lim_{n\to\infty} \left(\frac{n}{2}\right)^{2/n} = \lim_{n\to\infty} n^{2/n} \left(\frac{1}{2}\right)^{2/n} = L^2$$

by the product rule.

$$\lim_{n\to\infty} a^{1/n} = 1 \quad \forall a > 0$$
$$\left(a = \tfrac{1}{4}\right)$$

Subsequence $\{(2n/2)^{2/2n}\}$ of $\{(n/2)^{2/n}\}$ also converges to L^2. But $(2n/2)^{2/2n} = n^{1/n}$. Hence, $L^2 = L$ or $L(L-1) = 0$ and $L = 1$. $\square$

Series

Definition

Given a sequence $\{a_k\}$ of real or complex numbers, the formal expression

$$\sum_{k=1}^{\infty} a_k := a_1 + \cdots + a_k + a_{k+1} + \ldots$$

is called an (*infinite*) *series*. The numbers a_k are called the *terms* of the series. The finite sums

$$s_n := \sum_{k=1}^{n} a_k := a_1 + a_2 + \cdots + a_n$$

are called *partial sums*. They form a sequence $\{s_n\}$.

The Sum of a Series

Definition

An infinite series $\sum_{k=1}^{\infty} a_k$ *converges* if $\lim_{n\to\infty} s_n$ exists, and we define the *sum* of the series to be $s = \lim_{n\to\infty} s_n$. If $\{s_n\}\cancel{\}}$ does not converge, then we say the series *diverges*.

Example

If the real number $x > 0$ has decimal representation $a_0.a_1a_2\ldots$, then the series $\sum_{k=0}^{\infty} a_k 10^{-k}$ converges with sum x. Indeed the sequence $s_n = \sum_{k=0}^{n} a_k 10^{-k}$ is increasing and bounded above by x and $\lim_{n\to\infty} s_n = \sup\{s_n\} = x$.

Example

$$\sum_{k=\cancel{1}\,0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

$$s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} \to \frac{1-0}{1-\frac{1}{2}} = 2$$

Theorem

An infinite series $\sum_{k=1}^{\infty} a_k$ *of real or complex numbers converges if and only if for every* $\varepsilon > 0$ *there is an integer* $N = N(\varepsilon)$ *such that if* $m > n \geq N$*, then*

$$|s_m - s_n| = |a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon.$$

Proof.

$\sum a_k$ converges $\Leftrightarrow$ $\{S_n\}$ converges

$\Leftrightarrow$ $\{S_n\}$ satisfies Cauchy criterion

$\square$

The Harmonic Series

Example (The harmonic series)

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Does is converge? Look at partial sums:

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$$2^k + p \leq 2^{k+1} \quad \Downarrow \quad \frac{1}{2^k+p} \geq \frac{1}{2^{k+1}}$$

and their differences $s_m - s_n$ for $n = 2^k$ and $m = 2^{k+1} = 2^k + 2^k$:

$$p \leq 2^k \Rightarrow \frac{1}{2^k+p} \geq \frac{1}{2^{k+1}}$$

$$|s_m - s_n| = s_m - s_n = \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^k+2^k} \geq 2^k \frac{1}{2^{k+1}} = \frac{1}{2}.$$

Then Cauchy $\Rightarrow$ the series diverges and therefore the partial sums are unbounded. $\Rightarrow \sum \frac{1}{k}$ diverges to ∞

The nth Term Divergence Test

Theorem

If the series $\sum_{k=1}^{\infty} a_k$ *converges, then* $\lim_{k\to\infty} a_k = 0$.

Proof.

$\sum a_k$ converges $\Leftrightarrow$ Cauchy criterion

$\Leftrightarrow \forall \varepsilon > 0 \ \exists N : \forall n \geq N \ |S_{n+1} - S_n| = |a_{n+1}|$

$< \varepsilon \Leftrightarrow \lim_{n\to\infty} |a_{n+1}| = 0 \Leftrightarrow \lim_{n\to\infty} a_{n+1} = 0$

$\Leftrightarrow \lim_{n\to\infty} a_n = 0$ ↯

$\square$

Example

The series $\sum_{k=1}^{\infty} k\cos(1/k)$ diverges. $\left(k \cos\frac{1}{k} \to \infty \right)$

$\sum_{k=1}^{\infty} \frac{k-3}{2k}$ diverges $\left(\frac{k-3}{2k} \to \frac{1}{2} \right)$ b/c $\cos\frac{1}{k} \geq \frac{1}{2}$ for $k \geq 1$ as $\frac{1}{k} \leq \frac{\pi}{3}$

Sums of Series. Geometric Series

Theorem (Sums and Constant Multiples of Series)

If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then their sum, difference, and constant multiple by $c \in \mathbb{R}$ or $\mathbb{C}$ also converge; moreover, $\sum_{k=1}^{\infty}(a_k \pm b_k) = \sum_{k=1} a_k \pm \sum_{k=1} b_k$ and $\sum_{k=1} ca_k = c\sum_{k=1} a_k$.

Theorem (Geometric Series)

The geometric series $\sum_{k=0}^{\infty} q^k$ converges if $|q| < 1$ and diverges if $|q| \geq 1$. If $|q| < 1$, then

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

Proof.

$$S_n = \frac{1-q^{n+1}}{1-q} \rightarrow \frac{1-0}{1-q} \text{ if } |q|<1$$

$$|q| \geq 1 \Rightarrow |q^n| \geq 1$$

apply divergence test

The Euler number as a series

Theorem

The number $e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ *is the sum of the following series:*

$$e = \sum_{k=0}^{\infty}\frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \dots$$

Proof.

Series Converges : $\frac{1}{k!} \leq \frac{1}{2^k} \quad \forall\, k \geq 4 \ (k! \geq 2^k)$

$S_n = \sum_{k=0}^{n} \frac{1}{k!} \leq \sum_{k=0}^{n} \frac{1}{2^k} \leq 2$

Want $\lim_{n\to\infty} S_n = \lim_{n\to\infty} b_n$, $b_n = \left(1+\frac{1}{n}\right)^n$

step 1 $b_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\left(\frac{1}{n}\right)^k = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) +$

$$+\frac{1}{3!}\frac{n(n-1)(n-2)}{n^3}+\cdots+\frac{1}{n!}\frac{n(n-1)\cdots 1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=s_n$$

$$\Rightarrow e \stackrel{\text{def}}{=} \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} s_n.$$ Now let's see

if $e=\lim_{n\to\infty} b_n \geq \lim_{n\to\infty} s_n$ (next slide)

The Euler Number As a Series, continued

Step 2: Show $e = \lim_{n\to\infty} b_n \geq \lim_{n\to\infty} s_n$.

$$b_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{n!}\prod_{k=1}^{n-1}\left(1-\frac{k}{n}\right)$$

drop a few terms

$$\geq 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{m!}\prod_{k=1}^{m-1}\left(1-\frac{k}{n}\right) \quad \forall\, m \leq n$$

For any fixed m, let $n \to \infty$. Then $e = \lim_{n\to\infty} b_n$

$$\geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{m!} = s_m \quad \forall\, m \geq 1. \quad \square$$