Math 5615 Honors: Alternating Series and Absolute Convergence

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Alternating Series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - \dots \text{ or } \sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + \dots,$$

where $a_k > 0$ for all $k \geq 1$.

Theorem

If $\{a_k\}$ is a decreasing sequence of positive numbers such that $\lim_{k\to\infty} a_k = 0$, then the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ and $\sum_{k=1}^{\infty}(-1)^{k}a_k$ converge. For either of these convergent series, if s is the sum and s_n is the partial sum of the first n terms, then $|s_n - s| < |a_{n+1}|$. $= a_{n+1}$

Proof.

$a_1 - a_2 + a_3 - \dots$ (enough to consider only this)

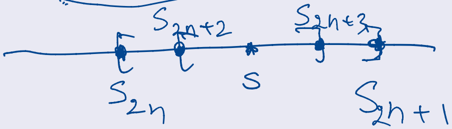
$0 < a_{k+1} \leq a_k \quad \forall k > 0 \quad k \in \mathbb{N}$

$\forall n \quad S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2} \geq S_{2n}$

Proof, Continued

$\forall n \quad S_{2n+3} = S_{2n+1} - a_{2n+2} + a_{2n+3} \le S_{2n+1}$

$$S_{2n+1} = S_{2n} + a_{2n+1} > S_{2n}$$



$$[S_2, S_3] \supset [S_4, S_5] \supset [S_6, S_7] \supset \dots$$

nested intervals

$$|[S_{2n}, S_{2n+1}]| = |S_{2n+1} - S_{2n}| = a_{2n+1}$$

$\longrightarrow 0$ as $n \to \infty \Rightarrow \bigcap_{n \ge 1} [S_{2n}, S_{2n+1}] = \{S\}$

Proof, Continued

$\lim_{n\to\infty} s_{2n} = s = \lim_{n\to\infty} s_{2n+1}$

Lemma $\Rightarrow \lim_{n\to\infty} s_n = s$, ~~b/c~~ <u>Proof of Lemma:</u>

$\forall \varepsilon > 0 \ \exists N_1 : \forall n \geq N_1 \ |s_{2n} - s| < \varepsilon$

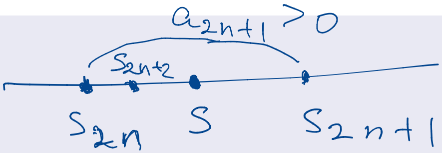
$\exists N_2 : \forall n \geq N_2 \quad |s_{2n+1} - s| < \varepsilon$

Take $N = 2\max(N_1, N_2) + 1$. Then

$\forall n \geq N \quad |s_n - s| = \begin{cases} |s_{2m} - s| < \varepsilon, \text{ b/c } m \geq N_1 & \text{if } n = 2m \\ |s_{2m+1} - s| < \varepsilon, \text{ b/c} & \text{if } n = 2m+1 \text{ then } m \geq N_2 \end{cases}$

either way $|s_n - s| < \varepsilon$. <u>Lemma proven</u>

Proof, Continued



strict actually, b/c otherwise $\{s_{2n+1}\}$ or $\{s_{2n}\}$ and $\{a_n\}$ are eventually constant

$$|s_n - s| = \begin{cases} |s_{2m} - s| \le |s_{2m+1} - s_{2m}| = a_{2m+1} & \text{if } n = 2m \\ |s_{2m+1} - s| \le |s_{2m+1} - s_{2m+2}| & \\ \quad = |a_{2m+2}| = a_{2m+2} & \text{if } n = 2m+1 \end{cases}$$

$$\Rightarrow |s_n - s| < a_{n+1}. \quad \Box$$

Alternating ~~Geometric~~ Harmonic Series. Absolute Convergence

Example

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Theorem applies, and this alternating series converges.

Definition

Let a_k be real or complex numbers. The series

$$\sum_{k=1}^{\infty} a_k$$

converges absolutely (is *absolutely convergent*) if the series with nonnegative terms $\sum_{k=1}^{\infty}|a_k|$ converges. The series $\sum_{k=1}^{\infty} a_k$ is said to be *conditionally convergent* if it converges, but $\sum_{k=1}^{\infty}|a_k|$ does not converge.

Absolute Convergence Implies Convergence

Theorem

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then it converges.

(The converse is not true:

Example

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$ converges conditionally

Proof.

Idea: use Cauchy criterion

$\sum |a_k|$ converges, want $\sum a_k$ to converge

$|S_m - S_n| = |a_{n+1} + a_{n+2} + \dots + a_m|$

$m > n$

$\leq |a_{n+1}| + \dots + |a_m| = |\tilde{S}_m - \tilde{S}_n|,$

Proof, Continued

where $\left\{ \begin{array}{l} s_n = a_1 + a_2 + \dots + a_n, \\ s'_n := |a_1| + |a_2| + \dots + |a_n|. \end{array} \right.$

Whenever $|s'_m - s'_n| < \varepsilon$,

we'll have $|s_m - s_n| < \varepsilon$.

Apply Cauchy's criterion twice: first to $\{s'_n\}$, then to $\{s_n\}$. $\square$

The Comparison Test

Theorem

Let $a_k \geq 0$, $c_k \geq 0$ and $d_k \geq 0$ for each k. The following statements are true:

1. *If there is an N such that $a_k \leq c_k$ for all $k \geq N$, and $\sum_{k=1}^{\infty} c_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If there is an N such that $a_k \geq d_k$ for all ~~kgeN~~ $k \geq N$, and $\sum_{k=1}^{\infty} d_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Proof.

$s_n' = a_N + a_{N+1} + \ldots + a_{N+m}$ is increasing

Its convergence $\Leftrightarrow$ boundedness. Then $\exists B_1 : \forall n \geq N$

$s_n' = a_N + a_{N+1} + \ldots + a_{N+n} \leq c_N + c_{N+1} + \ldots + c_{N+n} \leq B_1$

$\forall B_2 \exists n \geq N$:

$s_n' = a_N + a_{N+1} + \ldots + a_{N+m} \geq d_N + d_{N+1} + \ldots + d_{N+n} > B_2$

Examples

$$\sum_{k=1}^{\infty} \frac{5}{2^k + 3}, \qquad \sum_{k=1}^{\infty} \frac{k}{k^2 + k + 3}$$

convergent $\left(\leq \frac{5}{2^k} \right)$

divergent $\left(\geq \frac{k}{3k^2} = \frac{1}{3k} \text{ for } k \geq 2 \right)$