Math 5615 Honors: Limit superior and limit inferior

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Limit superior and limit inferior

Subsequential limits: (limits of subsequences)

$$\{a_n\} := \left\{1, -1, 1, -\frac{1}{2}, 1, -\frac{1}{3}, 1, -\frac{1}{4}, 1, \ldots\right\}?$$

$\{a_{2n-1}\} = \{1\}$
$\lim_{n\to\infty} a_{2n-1} = 1$

$\{a_{2n}\} = \{-1/n\} \to 0$

Limit points of the range?

Range: $\{1, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$

limit pts of it: $\{0\}$. A limit pt is a subsequential limit and another $p \in \mathbb{R}$ can be a subsequent. limit, iff $\exists$ ∞ many terms equal to p.

Infimum and supremum of the range?

$\inf = -1$, $\sup = 1$

Definition

1. The *limit superior* of $\{a_k\}$ is defined by

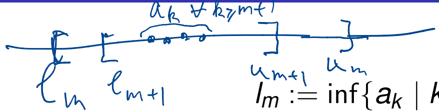
$$\limsup a_k := \inf_m \sup\{a_k \mid k \geq m\}.$$

2. The *limit inferior* of $\{a_k\}$ is defined by

$$\liminf a_k := \sup_m \inf\{a_k \mid k \geq m\}.$$

Here: Allow sup, inf to take values in $\mathbb{R} \cup \{\pm\infty\}$

Lim inf and lim sup, Continued



$l_1 \le l_m \le l_{m+1} \le u_{m+1} \le u_m \le u_1$

$$l_m := \inf\{a_k \mid k \ge m\}$$ increasing sequence

$$u_m := \sup\{a_k \mid k \ge m\}$$ decreasing sequence

$[l_m, u_m]$ nested intervals

$\liminf a_k = \sup l_m = \lim l_m = \lim_{m \to \infty} \inf\{a_k \mid k \ge m\}$

$\limsup a_k = \inf u_m = \lim u_m = \lim \sup\{a_k \mid k \ge m\}$

Example: lim inf and lim sup for the above sequence? ~~Observe:~~

$l_{2n} = -1/n$, $\Rightarrow \liminf a_k = \lim(-1/n) = 0$, $u_m = 1 \Rightarrow \limsup a_k = 1$

Note: 1. Subsequential limits of $\{a_k\}$ belong to $[\liminf a_k, \limsup a_k]$.

2. $\liminf a_k = \limsup a_k = L \iff \lim_{k \to \infty} a_k = L$, $L \in \mathbb{R} \cup \{\pm\infty\}$.

Theorem (See HW 6)

Let $\{a_k\}$ be a ~~bounded~~ sequence of real numbers and S is the set of all subsequential limits of $\{a_k\}$. Then

$$\sup S = \limsup a_k \text{ and } \inf S = \liminf a_k.$$

(s_k diverges to ∞)

Def. $\lim_{k\to\infty} s_k = \infty$ if $\forall M (>0) \in \mathbb{R}$ $\exists N \in \mathbb{N} : \forall n \geq N \quad s_n > M$.

(Idea: think of $\frac{1}{M}$ ∞ (M, ∞) as an open ball around ∞)

(s_k diverges to $-\infty$)

$\lim s_k = -\infty$ if $\forall M \in \mathbb{R}$ $\exists N \in \mathbb{N} : \forall n \geq N \quad s_n < M$.

(Even should allow $s_k = +\infty$ or $-\infty$ for the previous discussion to make sense)

$\lim s_k = a \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \forall n \geq N$ $|s_n - a| < \varepsilon$