Math 5615 Honors: Limits superior and inferior, continued
The root and ratio tests

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October 28, 2020

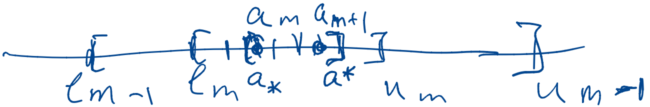
Lim inf and lim sup, Continued

$a_* := \liminf a_k = \lim_{m\to\infty} l_m = \inf\{l_m\}$; $a^* := \limsup a_k = \lim_{m\to\infty} u_m = \inf\{u_m\}$

$$l_m := \inf\{a_k \mid k \geq m\}$$

$$u_m := \sup\{a_k \mid k \geq m\}$$

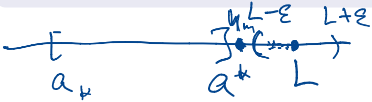
$l_{m-1} \leq l_m \leq u_m \leq u_{m-1} \Rightarrow [l_m, u_m]$ nested intervals



Proposition

1. Subsequential limits of $\{a_k\}$ belong to $[\liminf a_k, \limsup a_k]$.
2. For $L \in \mathbb{R} \cup \{\pm\infty\}$, $\lim a_k = L \Leftrightarrow \liminf a_k = \limsup a_k = L$.

1.



Suppose $\{a_{n_k}\} \to L > a^*$ ⇒ subseq limit $\leq \limsup a_k$

Choose $\varepsilon > 0$: $L - \varepsilon > a^*$

Then ∞ many terms of $\{a_{n_k}\}$ will be $> L-\varepsilon$

~~Ex~~ But outside of $[l_m, u_m]$ there are only $< \infty$ terms of $\{a_n\}$

Relation to Subsequential Limits

2. $\Rightarrow$ $\lim a_n = L$ $(L \in \mathbb{R})$

$\forall \varepsilon > 0 \; \exists N : \forall n > N \quad a_n \in (L - \varepsilon/2, L + \varepsilon/2)$

$L-\varepsilon \quad L-\varepsilon/2 \quad L \quad L+\varepsilon/2 \quad L+\varepsilon$

$L = -\infty$
$a_n < M$
$l_n, u_n \le M$

$\Rightarrow l_n, u_n \in [L - \varepsilon/2, L + \varepsilon/2] \quad \forall n > N$

$\Rightarrow l_n, u_n \in (L - \varepsilon, L + \varepsilon) \quad \forall n \ge N \Rightarrow \lim l_n = L$, $\lim u_n = L$

$\Leftarrow$ $\liminf a_n = L = \limsup a_n$, $L \in \mathbb{R}$

$\forall \varepsilon > 0 \; \exists N \; \forall n \ge N \quad l_n, u_n \in (L - \varepsilon, L + \varepsilon)$

$\Rightarrow \{a_n, a_{n+1}, a_{n+2}, \dots\} \subset (L - \varepsilon, L + \varepsilon) \Rightarrow \lim_{n \to \infty} a_n = L$

Theorem (See HW 6)

Let $\{a_k\}$ be a sequence of real numbers and S is the set of all subsequential limits of $\{a_k\}$, including $\pm\infty$. Then

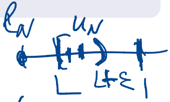
$$\sup S = \limsup a_k \text{ and } \inf S = \liminf a_k.$$

The Root Test

Theorem

Let $\sum a_k$ be a series of real numbers.
1. If $\limsup |a_k|^{1/k} < 1$, then the series converges absolutely.
2. If $\limsup |a_k|^{1/k} > 1$, then the series diverges.

Proof. 1. $0 \le L = \limsup |a_k|^{1/k} < 1$



Take $\varepsilon > 0$: $L + \varepsilon < 1$. Then $\exists N: \forall k \ge N$

$$|a_k|^{1/k} < L + \varepsilon \Rightarrow |a_k| < (L+\varepsilon)^k$$

$L + \varepsilon < 1$ $\quad \sum_{k \ge N} (L+\varepsilon)^k$ converges $\overset{\text{Comparison Test}}{\Longrightarrow}$ $\sum |a_k|$ converges

$\Rightarrow \sum a_k$ converges absolutely

The Proof of Root Test, Continued

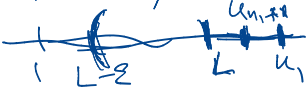
2. $L = \limsup |a_k|^{1/k} > 1$. Take $\varepsilon > 0$: $L - \varepsilon > 1$

(if $L = \infty$, take $M > 1$ instead of $L - \varepsilon$)

Claim: $\exists$ subsequence $\{|a_{n_k}|^{1/n_k}\}$: $|a_{n_k}|^{1/n_k} > L - \varepsilon$.

Indeed, $\exists n_1$: $|a_{n_1}|^{1/n_1} > L - \varepsilon$

$u_m = \sup_{k \ge m} |a_k|^{1/k}$



Then $u_{n_1+1} \ge L > L - \varepsilon$. Hence, $\exists n_2 > n_1$: $|a_{n_2}|^{1/n_2} > L - \varepsilon$.

... Claim is proven.

Thus, $|a_{n_k}|^{1/n_k} > L - \varepsilon > 1 \Rightarrow |a_{n_k}| > 1$

$\Rightarrow \lim_{k \to \infty} a_k \neq 0 \Rightarrow$ By nth term divergence test, $\sum a_k$ diverges. $\square$

The Ratio Test

Theorem

Let $\sum a_k$ *be a series of nonzero real numbers.*

1. If $\limsup |a_{k+1}/a_k| < 1$*, then the series converges absolutely.*

2. If $\liminf |a_{k+1}/a_k| > 1$*, then the series diverges.*

Proof. See textbook

Relation between Root and Ratio Tests

One can prove that

$$\liminf \left|\frac{a_{k+1}}{a_k}\right| \le \liminf |a_k|^{1/k} \le \limsup |a_k|^{1/k}$$

↑ we know just this

and

$$\limsup |a_k|^{1/k} \le \limsup \left|\frac{a_{k+1}}{a_k}\right|$$

$\Rightarrow$ Root test is stronger than Ratio Test