Math 5615 Honors: The ratio test: example
Dirichlet's test

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Example

For the series

$$1 + \frac{2}{3} + \frac{1}{3} + \frac{2}{3^2} + \frac{1}{3^2} + \frac{2}{3^3} + \frac{1}{3^3} + \dots,$$

we have $|a_{k+1}|/|a_k| = 2/3$ if k is odd and $|a_{k+1}|/|a_k| = 1/2\cancel{3}$ if k is even…

$$\lim_{k\to\infty} |a_{k+1}|/|a_k| = \text{doesn't exist}$$

$$\limsup |a_{k+1}|/|a_k| = \frac{2}{3} < 1$$

Converges by the ratio test.

Summation by parts

The root and ratio tests work well to show absolute convergence. Then next convergence test works well for conditionally convergent series.

Theorem (Abel's Summation by Parts)

Let $\{a_k\}_0^\infty$ and $\{b_k\}_0^\infty$ be sequences of real or complex numbers. For any integer $n \geq 0$, let

$$s_n(a) := \sum_{j=0}^{n} a_j.$$

Then

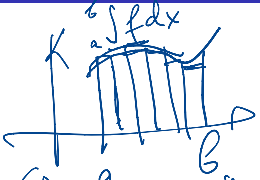
$\int df\, g \quad [f\, g]_0^n \quad \int f \quad dg$

$$\sum_{k=0}^{n} a_k b_k = s_n(a) b_{n+1} - \sum_{k=0}^{n} s_k(a)(b_{k+1} - b_k).$$

The Proof of Abel's Summation by Parts Theorem

Cf. $$\int_a^b g\,df = \left[fg\right]_a^b - \int_a^b f\,dg.$$

$[fg]_a^b := (fg)(b) - (fg)(a)$



($\int$ like a series)

Proof.

$S_n := S_n(a)$, $S_{-1} := 0$

$\forall k \geq 0 \quad a_k = S_k - S_{k-1}$

$$\sum_{k=0}^{n} a_k b_k = \sum_{k=0}^{n} (S_k - S_{k-1}) b_k = \sum_{k=0}^{n} S_k b_k - \sum_{k=0}^{n} S_{k-1} b_k$$

$S_{-1}=0$: $$= \sum_{k=0}^{n} S_k b_k - \sum_{k=1}^{n} S_{k-1} b_k = \sum_{k=0}^{n} S_k b_k - \sum_{k=0}^{n-1} S_k b_{k+1}$$

$$= \sum_{k=0}^{n} S_k b_k - \sum_{k=0}^{n} S_k b_{k+1} + S_n b_{n+1} = \sum_{k=0}^{n} S_k (b_k - b_{k+1}) + S_n b_{n+1} \quad \square$$

Dirichlet's Test

Theorem

The series $\sum_{k=0}^{\infty} a_k b_k$ converges if the following conditions hold:
1. The partial sums $s_n(a) = \sum_{k=0}^{n} a_k$ form a bounded sequence;
2. $b_0 \geq b_1 \geq b_2 \geq \ldots$;
3. $\lim_{k\to\infty} b_k = 0$.

Proof.

$$\sum_{k=0}^{n} \left| |s_k(a)| \cdot |(b_{k+1} - b_k)| \right| \leq M \sum_{k=0}^{n} |b_{k+1} - b_k|$$

$$= M \sum_{k=0}^{n} (b_k - b_{k+1}) = M(b_0 - \cancel{b_1} + \cancel{b_1} - \cancel{b_2} + \ldots + \cancel{b_n} - b_{n+1}) = M(b_0 - b_{n+1}) \leq M' \quad b/c \ \{b_n\} \text{ converges} \Rightarrow \{b_n\} \text{ bdd}$$

Thus $\sum_{k=0}^{n} |s_k(a)|(b_{k+1} - b_k)$ are bdd $\Rightarrow \sum_{k=0}^{\infty} s_k(a)(b_{k+1} - b_k)$ converges

The Proof of Dirichlet's Theorem, Continued

Proof.

$$0 \le |S_n(a)\, b_{n+1}| \le M\, |b_{n+1}| \to 0$$

$$\Rightarrow \lim_{n\to\infty} |S_n(a)\, b_{n+1}| = 0 \Rightarrow \lim_{n\to\infty} S_n(a)\, b_{n+1} = 0$$

By $\lim (c_n - d_n) = \lim c_n - \lim d_n$, if these exist

we get the conclusion. $\square$

Alternating Series

Corollary

If $\{a_k\}$ is monotone decreasing with limit 0, then the alternating series

$$\sum_{k=\cancel{1}\,0}^{\infty}(-1)^k a_k$$

converges.

Proof. Dirichlet's test $\Rightarrow$ $1-1+1-1+1-\ldots$

$\left.\begin{array}{l} \left\{ S_n((-1)^k) = \sum_{k=0}^{n}(-1)^k = \{1, 0\} \Rightarrow \text{bdd} \right. \\ \{a_n\} \text{ monotone decreasing to } 0 \end{array}\right\} \Rightarrow \sum(-1)^k a_k$ converges

$\square$