Math 5615H: Honors: Introduction to Analysis
The Cantor Set
Connected Sets
Sequences and Their Limits

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The Cantor Set: Uncountable Set of Measure Zero

Idea: extract the middle thirds from $[0, 1] \subset \mathbb{R}$:

D_2

$$D_0 = \varnothing,$$
$$D_1 = \left(\frac{1}{3}, \frac{2}{3}\right),$$
$$D_2 = \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right),$$
$$D_3 = \left(\frac{1}{27}, \frac{2}{27}\right) \cup \left(\frac{7}{27}, \frac{8}{27}\right) \cup \left(\frac{19}{27}, \frac{20}{27}\right) \cup \left(\frac{25}{27}, \frac{26}{27}\right),$$
$$\ldots$$

Each of the sets D_n is extracted, or carved away, from $[0, 1]$.

The Cantor Set, Continued

Definition

The *Cantor set* C is the complement in $[0, 1]$ of the union of the sets D_n:

$$C := \left\{ x \in [0,1] \mid x \notin \bigcup_{k=0}^{\infty} D_k \right\} = [0,1] \setminus \bigcup_{k=0}^{\infty} D_k .$$

Observe: $\quad C = \bigcap_{k=0}^{\infty} C_k,$ where

$$C_k = \left(\bigcup_{i=0}^{k} D_i \right)^c$$

$$C_0 = [0,1],$$

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],$$

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \ldots$$

How much space does the Cantor set take in [0,1]?

1. $\sum_{k\geq 1} \mu(D_k) =$

2. C may be covered by a finite number of closed intervals of arbitrarily small total length, see current HW.

(not necessarily disjoint intervals)

Ternary expansions of numbers x in $[0, 1]$

Repeat the same construction as for decimal and binary epansions, now base-3: After having $b_0.b_1 \ldots b_n \leq x$ constructed, take $b_{n+1} \geq 0$ the greatest so that

$$b_0.b_1 \ldots b_n b_{n+1} = \sum_{j=0}^{n+1} b_j/3^j \leq x.$$

(The fact that it is a ternary (base-3) expansion actually means

$$x = 0.b_1 b_2 \cdots = \sum_{j=1}^{\infty} b_j/3^j.)$$

$(q \in \mathbb{Z}, q \geq 0)$

But for numbers $q3^{-k}$, which are exactly those which expand as $b_0.b_1 b_k\overline{000}\ldots$, do a revision, today only:

$b_0.b_1 \ldots b_k 000\ldots$

$$(5)3^{-3} = \cancel{.011\overline{222}\cdots} = .012\overline{000}\ldots,$$
$$(4)3^{-3} = .010\overline{222}\cdots = \cancel{.011\overline{000}}\ldots$$

Use $\ldots 0\overline{222}\ldots$ instead of $\ldots 1\overline{000}\ldots$!

Thus, $1/3 = .1 = .0\overline{222}\ldots$, but $2/3 = .2\overline{000}\ldots$

$1 = 0.\overline{222}\ldots$

The Cantor Set Is Uncountable

Theorem (Midterm Exam 1)

The Cantor set C consists of all the numbers in the closed interval $[0,1]$ whose ternary expansion has only 0's and 2's and may end in infinitely many 2's:

$$C = \{x = 0.b_1 b_2 \cdots \in [0,1] \mid b_i = 0 \text{ or } 2\}.$$

Corollary

The Cantor set is uncountable.

Yet, C is very far from being dense, unlike uncountable $\mathbb{R} \setminus \mathbb{Q}$.

Suppose not, i.e., C countable. Then C is given by a list

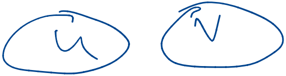
$$x_1 = 0.b_{11} b_{12} b_{13} \dots$$
$$x_2 = 0.b_{21} b_{22} b_{23} \dots$$
$$\vdots$$

$$x = 0.\bar{b}_{11} \bar{b}_{22} \bar{b}_{33} \dots \quad \text{where } \bar{b}_{nn} = \begin{cases} 0 & \text{if } b_{nn} = 2 \\ 2 & \text{if } b_{nn} = 0 \end{cases}$$

Then $x \neq x_n \ \forall n$

A disconnected metric space X: $X = U \cup V$, $U \cap V = \varnothing$, $U \neq \varnothing$, $V \neq \varnothing$, and U, V open.

A disconnected subset $S \subset X$: (S, d) is disconnected as metric space, i.e., $\exists\, U, V$ open $\subset X$, $U \cap S \neq \varnothing$, $V \cap S \neq \varnothing$,

$X =$ 

$(U \cap S) \cap (V \cap S) = \varnothing$

$S = (U \cap S) \cup (V \cap S)$

or, equivalently, $S \subset U \cup V$ in X.

$S \subseteq X$ connected, if it's not disconnected.

$\varnothing$, $\{x\}$ are connected

Intervals in $\mathbb{R}$

Def. $[a,b]$, $(a,b]$, $[a,b)$, (a,b)

$a,b \in \mathbb{R} \cup \{-\infty, \infty\}$

($[-\infty, 3)$ and alike not allowed)

HW Problem: $S \subset \mathbb{R}$ is an interval iff $\forall x, y \in S$, $x < y$, and $\forall z$: $x \le z \le y$, $z \in S$.

Connected Subsets of $\mathbb{R}$

Theorem

A subset S of $\mathbb{R}$ is connected iff it is an interval. In particular $\mathbb{R}$ is connected.

$\Rightarrow$: Suppose S is not an interval, i.e., $\exists x < y \in S$ and $z \notin S$ with $x < y < z$. Want to show S disconnected.

$\Leftarrow$: Suppose S is an interval but disconnected by open U and V of $\mathbb{R}$. Want: contradiction.

$$\mu(C) = 1 - \mu(D_0) - \mu(D_1) - \mu(D_2) - \dots$$

(Lebesgue) measure

$$= 1 - \left(0 + \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots\right) = 0$$

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \frac{1}{1-\frac{2}{3}} = 1$$

$$C = \left(\bigcup_{k \geq 0} D_k\right)^c \subset [0,1]$$

$$\mu(C) \overset{\text{expected}}{=} \mu([0,1]) - \sum_{k=0}^{\infty} \mu(D_k)$$