Math 5615H: Honors: Connected Sets in $\mathbb{R}$ Sequences and Their Limits

Sasha Voronov

University of Minnesota

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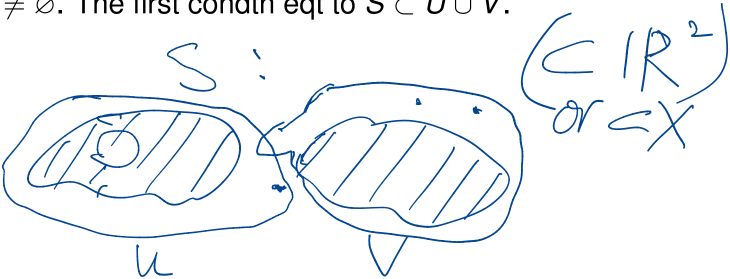
Connected Sets

A disconnected metric space X: $X = U \cup V$, $U \cap V = \varnothing$, $U \neq \varnothing$, $V \neq \varnothing$.
A disconnected subset $S \subset X$: (S, d) is disconnected as metric space. Equivalently, $\exists$ open $U, V \subset X$ s.t. $S = (S \cap U) \cup (S \cap V)$, $(S \cap U) \cap (S \cap V) = \varnothing$, $S \cap U \neq \varnothing$, $S \cap V \neq \varnothing$. The first condtn eqt to $S \subset U \cup V$.



Connected subset: one which is not disconnected.

Intervals in $\mathbb{R}$

$= \{z \in \mathbb{R} \mid a \le z < b\}$

$[a, b]$, $[a, b)$, $(-\infty, b]$, etc. $a \nleq b$ (not necessarily)

Not on current HW (simple exercise): $I \subset \mathbb{R}$ is an interval iff $\forall x, y \in I$ and $z \in \mathbb{R}$: $x \le \cancel{y}\, z \le \cancel{z}\, y \Rightarrow z \in I$. (or $x < z < y \Rightarrow z \in I$)

Trivial intervals: $\varnothing$ and singleton $\{a\} = [a, a]$. Other intervals will be called *nontrivial*.

Theorem

A subset S of $\mathbb{R}$ is connected iff it is an interval. In particular $\mathbb{R}$ is connected.

$\Rightarrow$: Suppose S is not an interval, i.e., $\exists x < y \in S$ and $z \notin S$ with $x < z < y$ Want to show S disconnected.

$U = (-\infty, z)$, $V = (z, \infty)$ open of $\mathbb{R}$

$U \cap V = \emptyset$, $S \subseteq U \cup V = \mathbb{R} \setminus \{z\}$, $S \cap U \ni x$, $S \cap V \ni y$

$\Leftarrow$: Suppose S is a **nontrivial** interval but disconnected by open U and V of $\mathbb{R}$. Want: contradiction. Let $x \in U \cap S$, $y \in V \cap S$.

WLOG: $x < y$. Idea: try to extend $S \cap U$ to the right of x as far as possible; $[x, z)$, $z > x$ – look at such intervals

Then $[x, \alpha)$ is the longest extn

$\alpha := \sup\{z \in \mathbb{R} \mid x < z \le y \text{ and } [x, z) \subset S \cap U\}$

Nonempty: x interior in $U \Rightarrow x \in (x-r, x+r) \subset U$, $r > 0$, $[x, x+r) \subset S$ if r is small enough

$[x, x+r) \subset S \cap U$

Continuation of proof: S an interval $\Rightarrow$ S connected

$x < \alpha \le y \Rightarrow \underline{\alpha \in S}$ (b/c S includes all intermed. pts)

~~If~~ Then $\alpha \in U$ or V.

(1) If $\alpha \in U$ (and thereby $\alpha \neq y$), since U open, $\exists\, \varepsilon_1 > 0$: $[\alpha, \alpha+\varepsilon_1) \subset U \cap S$ for ε_1 small enough

$\Rightarrow [x, \alpha+\varepsilon_1) \subset S \cap U$

contradicts (α is an upper bd)

(2) If $\alpha \in V$. Since V open, $\exists\, \varepsilon_2 > 0$: $(\alpha-\varepsilon_2, \alpha] \subset S \cap V$ if ε_2 is small enough

$\Rightarrow \alpha$ not the least upper bd; $\alpha - \varepsilon_2/2$ will be an upper bd.

Contradiction. $\square$

Corollary

The only subsets of $\mathbb{R}$ that are both open and closed are $\varnothing$ and $\mathbb{R}$.

$\emptyset \neq S \subsetneq \mathbb{R}$ both op. & closed

$\Rightarrow (S^c)^{\neq \emptyset}$ also open & closed

$\Rightarrow S \cup S^c$ disconnects $\mathbb{R}$

Definition (sequence as $a : \mathbb{N} \to X$ or $\{a_n\}$, limit, converges, convergent):

$a_n := a(n) \in X$

$X \ni L = \lim_{n\to\infty} a_n$ if $\forall \varepsilon > 0\ \exists \underline{N} \in \mathbb{N} : \forall n > N\ d(a_n, L) < \varepsilon.$

ε a_n $n > N$ L

Uniqueness:

$L_1, L_2 \in X$ $L_1 = \lim a_n$, $L_2 = \lim a_n \Rightarrow L_1 = L_2$

$L_1 \neq L_2$, $\varepsilon = d(L_1, L_2)/2 \Rightarrow L_1$ is not the limit of $\{a_n\}$ nor L_2

Example: $\{\frac{1}{n}\}$ sequence in $\mathbb{R}$ (numerical sequence)

$\lim \frac{1}{n} = 0$: $\forall \varepsilon > 0$ take $N \in \mathbb{N}$: $N > \frac{1}{\varepsilon}$ (Arch. property)

$\forall n > N$ $\left|\frac{1}{n}\right| < \frac{1}{N} < \varepsilon$

Boundedness: (Choose $\varepsilon = 1$) $\exists \lim a_n \Rightarrow \{a_n\}$ bdd

$d(a_n, a_m) \leq \max\left(d(x_i, x_j),\ d(x_i, L) + 1 \right)$ $x_1, \dots, x_i$ $< \infty$ many