# Math 5615 Honors: Further Topics on Continuity

Sasha Voronov

University of Minnesota

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**Theorem**

*If a function $f : K \to Y$ is continuous on a compact set $K \subset X$, then $f(K)$ is compact.*

**Proof.** Suppose $\{U_\alpha \mid \alpha \in I\}$ is an open cover of $f(K)$.
Let $V_\alpha \subset X$ be open such that $V_\alpha \cap K = f^{-1}(U_\alpha)$
($V_\alpha$ exists for each $\alpha$, b/c $f^{-1}(U_\alpha)$ is open rel. to $K$.)
Note that $\{V_\alpha \mid \alpha \in I\}$ form an open of $K$:
$K \subset \bigcup_{\alpha\in I} V_\alpha$: indeed $\forall x \in K$ $f(x) \in f(K)$, and $\Rightarrow$
$f(x) \in U_\alpha$ for some $\alpha$ $\Rightarrow x \in f^{-1}(U_\alpha) \subset V_\alpha$.
$K$ compact $\Rightarrow \exists \{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$, a finite subcover of $K$.
Claim: $\{U_{\alpha_i} \mid i = 1, \ldots, n\}$ is a finite subcover of $\{U_\alpha \mid \alpha \in I\}$. Indeed $f(K) \subset \bigcup_{i=1}^{n} U_{\alpha_i}$, b/c $\forall y \in f(K)$ $\exists x \in K$ $y = f(x)$. Then $x \in V_{\alpha_j}$ for some $j$, therefore, $f(x) = y \in U_{\alpha_j}$. $\square$

# Uniform Continuity

**Definition**

We say that $f$ is *uniformly continuous on $D$* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in D$ and $d(x, y) < \delta$, then $d(f(x), f(y)) < \varepsilon$.

Difference with continuity at each point $a \in D$ (roughly):
**Continuity on** $D$: $\delta = \delta(a, \varepsilon)$,
**Uniform continuity on** $D$: $\delta = \delta(\varepsilon)$.

Note: Unif. continuity on D implies continuity on D.

**Theorem**

*If a function $f : K \to Y$ is continuous on a compact set $K \subset X$, then $f$ is uniformly continuous.*

**Proof.** Suppose $f$ is not uniformly continuous on $K$. This means there is an $\varepsilon > 0$ such that for each $\delta > 0$, say, $\delta_n = 1/n$, there exist $x_n, z_n \in K$ such that $d(x_n, z_n) < 1/n$ but $d(f(x_n), f(z_n)) \geq \varepsilon$. This gives two sequences $\{x_n\}$ and $\{z_n\}$ such that $\lim_{n\to\infty} d(x_n, z_n) = 0$ but $d(f(x_n), f(z_n)) \geq \varepsilon$.

# Proof, Continued

$K$ is compact $\Rightarrow$ sequentially compact $\Rightarrow$ $\exists$ subsequence $\{x_{n_k}\} \to a \in K$. Since $d(x_{n_k}, z_{n_k}) \to 0$ as $k \to 0$. Note $\{z_{n_k}\} \to a$ as $k\to\infty$, b/c

$$d(a, z_{n_k}) \le d(a, x_{n_k}) + d(x_{n_k}, z_{n_k})$$ implies

$d(a, z_{n_k}) \to 0$, i.e., $\{z_{n_k}\} \to a$.

Sequential charactzn of continuity of $f(x)$ at $a$: get $f(x_{n_k}) \to f(a)$ and $f(z_{n_k}) \to f(a)$.

$\Rightarrow$ $d(f(x_{n_k}), f(a)) \to 0$ and $d(f(z_{n_k}), f(a)) \to 0$

$\Rightarrow 0 \le d(f(x_{n_k}), f(z_{n_k})) \le d(f(x_{n_k}), f(a)) + d(f(a), f(z_{n_k}))$

$\to 0$, which contradicts $d(f(x_{n_k}), f(z_{n_k})) \ge \varepsilon$ $\forall k$. $\square$

Examples. 1. $f(x) = x^2 : \mathbb{R} \to \mathbb{R}$ conts. is not unif. continuous. Take $\varepsilon = 1$; then $\forall \delta > 0$ $\exists x, y \in \mathbb{R} : |x - y| < \delta$ but $|x^2 - y^2| \geq 1$.

$|x^2 - y^2| = |x - y| \cdot |x + y|$. Take $x \geq \frac{1}{\delta}$ and $y = x + \frac{\delta}{2}$. Then $|x - y| = \delta/2 < \delta$ and $|x + y| = 2x + \frac{\delta}{2} > 2x \geq \frac{2}{\delta}$.

Then $|x^2 - y^2| = |x - y| \cdot |x + y| = \frac{\delta}{2} |x + y| > \frac{\delta}{2} \cdot \frac{2}{\delta} = 1$.

2. $f(x) = x^2$ on $[0, 1]$ is unif. conts by Thm.

3. $f(x) = \sin x$ on $\mathbb{R}$ is unif. conts. (The argument is a little more complicated than for $f(x) = x^2$)

# The Extreme Value Theorem

## Theorem

*A real-valued continuous function $f : K \to \mathbb{R}$ is continuous on a compact set $K \subset X$ achieves its absolute maximum and absolute minimum value on $K$; that is, there exist points $x_M$ and $x_m$ in $K$ such that $f(x_m) \leq f(x)$ for all $x \in K$, and $f(x_M) \geq f(x)$ for all $x \in K$.*

## Proof.

The image set $f(K)$ is a compact subset of $\mathbb{R}$, so $f(K)$ is bounded and closed, and hence $f(K)$ contains $\inf f(K)$ and $\sup f(K)$, because they have to be elements of $f(K)$ or its limit points. $\square$

## Example

$f(x) = 1/x$ is continuous on $(0, \infty)$ but does not achieve maximum and minimum values. It does on $[1, 2]$ or any closed interval within $(0, \infty)$. It does not on $[-1, 1]$.