Math 5615 Honors: Infinite Limits and Limits at Infinity
Discontinuities

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Limits at Infinity

Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let D be unbounded above for limit at ∞ (or below for limit at $-\infty$). Let $L \in \mathbb{R}$. We write

$$\lim_{x\to\infty} f(x) = L \qquad \left(\lim_{x\to-\infty} f(x) = L\right)$$

if for every $\varepsilon > 0$ there is an $M > 0$ such that ($x \in D$ and $x > M \quad$ (resp. $x < -M$)) $\Rightarrow |f(x) - L| < \varepsilon$.

Intuition: $\{x \in \mathbb{R} \mid x > M\}$ and $\{x \in \mathbb{R} \mid x < -M\}$ are "open balls" about ∞ and $-\infty$, resp. "Unbounded above domain D" is the replacement of "∞ is a cluster point of D."

Example

For $f(x) = 1/x$ defined on $(0, \infty)$, $\lim_{x\to\infty} f(x) = 0$. Indeed, for each ε, choose $M = 1/\varepsilon$. Then for each $x > M$

$|f(x)| = 1/x < 1/M = \varepsilon.$

Properties of Limits at Infinity

Limits $\lim_{x\to\pm\infty} f(x)$ have all the properties we have studied about limits, such as uniqueness, sums, products, squeeze, quotients (with necessary precautions).

Example (related to infinite limits on the next slide)

$$\lim_{x\to a} f(x) = L \in \mathbb{R}$$

$$\lim_{x\to a} g(x) = -\infty$$

$\boxed{f, g : D \to \mathbb{R},\ D \subset \mathbb{R},\ g(x) \neq 0\ \forall x \in D\ \boxed{\text{assumed}}\quad a \in D \text{ clusterpt}}$

$$\Downarrow$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0$$

Infinite Limits

Definition

Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let a be a cluster point of D. We say that $\lim_{x \to a} f(x) = \infty$ if for every $N > 0$ there is a $\delta = \delta(N) > 0$ such that if $x \in D$ and $0 < |x - a| < \delta$, then $f(x) > N$. Similarly, we say that $\lim_{x \to a} f(x) = -\infty$ if for every $N > 0$ there is a $\delta = \delta(N) > 0$ such that if $x \in D$ and $0 < |x - a| < \delta$, then $f(x) < -N$.

The notational convention of this definition can be extended to the case of $a = \pm\infty$ being a "limit point" of D. For example, $\lim_{x \to \infty} f(x) = \infty$ if for every $N > 0$ there is an $M = M(N) > 0$ such that if $x \in D$ and $x > M$, then $f(x) > N$.

Example

For $f(x) = 1/|x|$ defined on $\mathbb{R} \setminus \{0\}$, $\lim_{x \to 0} f(x) = \infty$. Indeed, for each $N > 0$, choose $\delta = 1/N$. Then $\forall\, x \neq 0 : |x| < \delta$ we have $f(x) = \frac{1}{|x|} > \frac{1}{\delta} = N$.

$y = 1/|x|$

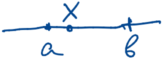
Discontinuities

$a \in X$ is a discontinuity of $f: D \subset X \to Y$ if f fails to be continuous at $x=a$.

Let $f : (a, b) \to \mathbb{R}$. We say that f has a *right-hand limit at a*, denoted $f(a+)$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$\mathbb{R} \ni$

$$0 < x - a < \delta \Rightarrow |f(x) - f(a+)| < \varepsilon.$$



Cf.: $0<|x-a|<\delta$ for $\lim_{x\to a} f(x)$

We write $f(a+) := \lim_{x \to a+} f(x)$ when this limit exists. $\in \mathbb{R}$
We say that f has a *left-hand limit at b*, denoted $f(b-)$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < b - x < \delta \Rightarrow |f(x) - f(b-)| < \varepsilon.$$

We write $f(b-) := \lim_{x \to b-} f(x)$ when this limit exists.
A function f is said to have a *discontinuity of the first kind, or simple discontinuity*, at the point a if the one-sided limits $f(a+)$ and $f(a-)$ both exist, but f is discontinuous at a. If they are unequal: $f(a+) \neq f(a-)$, we call it a *jump discontinuity*. If they are equal: $f(a+) = f(a-)$, we call it a *removable discontinuity*.

Discontinuities of the First Kind

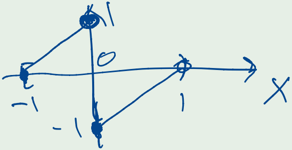
Examples

$$f(x) = \begin{cases} x+1, & \text{if } -1 \le x < 0, \\ x-1, & \text{if } 0 \le x < 1, \end{cases}$$

has a jump discontinuity at $x = 0$.

$f(0-) = \lim_{x \to 0-} f(x) = 1$

$f(0+) = \lim_{x \to 0+} f(x) = -1$



$$f(x) = \frac{\sin x}{x}, \qquad \text{if } x \neq 0,$$

has a removable discontinuity at $x = 0$.

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = \boxed{\lim_{x \to 0} f(x) = 1}$$

If we define $f(0) := 1$, the resulting funct. is cts at $x = 0$.

Discontinuities of the Second Kind

A function f is said to have a *discontinuity of the second kind* at the point a if either of the one-sided limits $f(a+)$ and $f(a-)$ fails to exist. A particular case: f has an *infinite discontinuity at a* if either of the one-sided limits at a is infinite.

Examples

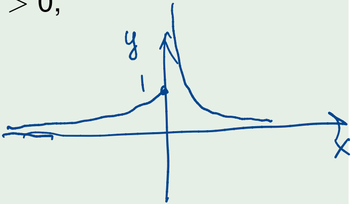
$$f(x) = \sin\frac{1}{x}, \qquad \text{if } x \neq 0,$$

has a discontinuity of the second kind at $x = 0$.

$f(a+), f(a-)$ don't exist

$$f(x) = \begin{cases} e^x, & \text{if } x \leq 0, \\ \frac{1}{x}, & \text{if } x > 0, \end{cases}$$

has an infinite discontinuity at $x = 0$.



$$\lim_{x \to 0-} f(x) = 1$$

$$\lim_{x \to 0+} f(x) = \infty$$