Math 5615 Honors: Discontinuities of Monotone Functions
The Derivative

Sasha Voronov

University of Minnesota

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Reminder: Classification of Discontinuities

Discontinuities of $f:(a,b)\to\mathbb{R}$, $c\in(a,b)$

- 1st kind, or simple: both $f(c-)$, $f(c+)$ exist
  - removable: $f(c-)=f(c+)$
  - jump: $f(c-)\neq f(c+)$
- 2nd kind (otherwise)
  - infinite: $f(c-)=\pm\infty$ or $f(c+)=\pm\infty$
  - other

Discontinuities of Monotone Functions

Definition

Let $f : I \to \mathbb{R}$ where I is an interval. Then
1. f is *monotone* (*strictly*) *increasing on* I if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$, $(f(x_1) < f(x_2)$, resp.);
2. f is *monotone* (*strictly*) *decreasing on* I if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$ $(f(x_1) > f(x_2)$, resp.).

Theorem

Monotone functions on an open interval I have discontinuities only of the first kind, more specifically, jump discontinuities.

Countability of the set of discontinuities of a monotone function

Corollary

A monotone function on an open interval I has at most countably many discontinuities.

WLOG assume f is increasing

Proof of Corollary. For each discontinuity, choose a rational number in the "jump interval." This gives an injective map from the set of discontinuities in I to $\mathbb{Q}$.

$\forall x \in I$ at which f is not continuous:

Choose $f(x-) < r(x) < f(x+)$, $r(x) \in \mathbb{Q}$

Get $r: E \to \mathbb{Q}$, where $E \subset I$ is the set of discontinuities.

Claim: r is injective. If $x_1 < x_2 \in E$, then $f(x_1+) \leq f(x_2-)$ and $\Rightarrow r(x_1) < r(x_2)$.

Thus, E is at most countable. $\square$

Proof of Theorem on Discontinuities of a Monotone Function

WLOG assume f is monotone increasing. Let $a \in I$. Then $f(x) \leq f(a)$ for all $x < a$. Thus, the set

$$L := \{f(x) \mid x < a\}$$

is bounded above. Let $M := \sup L$. If $\varepsilon > 0$, then there exists $p \in I$ such that $p < a$ and

$$0 \leq M - f(p) < \varepsilon.$$

Otherwise, if $\forall p < a$, $M - f(p) \geq \varepsilon$, then $M - \varepsilon$ would be a smaller upper bound for L.

$f(p)$ $f(x)$ M

Then whenever $p < x < a$, we have $f(p) \leq f(x) \leq M$ and so

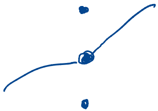
$$0 \leq M - f(x) < \varepsilon.$$

(I.e., use $\delta := a - p$.)

Therefore, $f(a-) = \lim_{x \to a-} f(x)$ exists and equals M.

Proof of Theorem, Concluded

Right-hand limit: The set $R := \{f(x) \mid x > a\}$ is bounded below, let $m := \inf R$. Then, similarly, $\lim_{x \to a+} f(x) = m$.

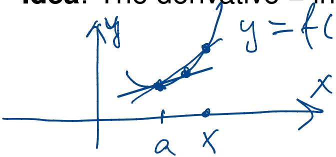


Jump, rather than removable, *i.e.*, $m \overset{>}{\cancel{<}} M$:

Indeed, if $M = f(a-) = f(a+) = m$, then since
$f(a-) \le f(a) \le f(a+)$, we have
$f(a-) = f(a) = f(a+)$, whence
$\lim_{x \to a} f(x)$ exists and equals $f(a)$;
i.e., f is conts at a, $\square$

The Derivative: Definition

Idea: The derivative = instantaneous slope of the chord at $x=a$



$y = f(x)$

$(a, f(a))$, $(x, f(x))$

slope of chord $= \dfrac{f(x)-f(a)}{x-a}$

Definition

Let I be an interval of real numbers, let $f : I \to \mathbb{R}$, and suppose $a \in I$ is an interior point. If the limit

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$\cancel{\dfrac{f(a)-f(a)}{a-a}}$ not defined

exists, then f is said to be *differentiable at a*, and the limiting value is denoted by $f'(a)$ and called the *derivative of f at a*. If I is an open interval, and if f is differentiable at every $a \in I$, then we say f is *differentiable on I*.

The Derivative: Properties

Remark.

$$\frac{f(x)-f(a)}{x-a} = \frac{f(a+h)-f(a)}{h},$$

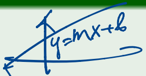
if we set $h := x - a$. Also, $0<|h| < \delta \Leftrightarrow 0<|x-a| < \delta$. We say this means $h \to 0 \Leftrightarrow x \to a$ and

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a},$$

meaning: when one of the limits exists, the other exists and equals the first one.

Examples

1. $f(x) = mx + b$, $m, b \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$.
2. $f(x) = |x|$.



1. $\lim_{x\to a}\frac{mx+b-(ma+b)}{x-a} = \lim_{x\to a}\left(m\cdot\frac{x-a}{x-a}\right) = m\cdot 1 = m$

2. $\lim_{x\to 0}\frac{|x|-0}{x} = \lim_{x\to 0}\begin{cases}+1, & x>0,\\ -1, & x<0,\end{cases}$ doesn't exist