Math 5615 Honors:The Derivative

Sasha Voronov

University of Minnesota

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Reminder: The Derivative

Definition

Let I be an interval of real numbers, let $f : I \to \mathbb{R}$, and suppose $a \in I$ is an interior point. If the limit

$$\frac{df}{dx}(a) := f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

exists, then f is said to be *differentiable at a*, and the limiting value is denoted by $f'(a)$ and called the *derivative of f at a*. If I is an open interval, and if f is differentiable at every $a \in I$, then we say f is *differentiable on I*.

Differentiability and Continuity

Theorem

Let $f : I \to \mathbb{R}$ and let $a \in I$ be an interior point of I. If $f'(a)$ exists, then f is continuous at a.

Proof.

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

$\forall x \in I,\ x \neq a$. Pass to limit as $x \to a$:

$$\lim_{x \to a} \left(f(x) - f(a) \right) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a)$$

$$= f'(a) \cdot 0 = 0 \Rightarrow f(x) \text{ is}$$

continuous at a. $\square$

Continuity and Differentiability, Continued

Differentiability at a points is strictly stronger than continuity at a point: e.g., $f(x) = |x|$. $|x|'(0)$ doesn't exist, but $|x|$ is cont's at 0.

Example

If g is differentiable at x and $g(x) \neq 0$, then by continuity at x $g(x+h) \neq 0$ for $|h|$ suff. small

$$\left(\frac{1}{g(x)}\right)' = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \cdot \frac{g(x) - g(x+h)}{h}$$

$$= \lim_{h \to 0} \frac{1}{g(x+h)\,g(x)} \cdot \left(-\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right)$$

$$= \frac{1}{g(x)^2}\left(-g'(x)\right) = -\frac{g'(x)}{(g(x))^2}$$

by continuity of $g(x)$

Concrete Exampe: $\cot' x$

Example

Let us compute $\cot' x$ using the derivative $\tan' x = \sec^2 x = 1/\cos^2 x$ of $\tan x$:

$$\cot' x = \left(\frac{1}{\tan x}\right)' = -\frac{\tan' x}{(\tan x)^2}$$

$$= -\frac{\sec^2 x}{\tan^2 x} = \frac{-1/\cos^2 x}{\sin^2 x/\cos^2 x} = -\frac{1}{\sin^2 x}$$

$$= -\csc^2 x$$

The Product and Quotient Rules

Theorem

Suppose f and g are real valued functions defined in an open interval about a. If f and g are both differentiable at a, then

1. *$f \pm g$ is differentiable at a and $(f \pm g)'(a) = f'(a) \pm g'(a)$;*
2. *The product function $(fg)(x) := f(x)g(x)$ is differentiable at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$;*
3. *If $g(a) \neq 0$, the quotient function $(f/g)(x) := f(x)/g(x)$ is differentiable at a and $(f/g)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$.*

Proof: The product rule.

$$\frac{f(x)g(x) - f(a)g(a)}{x-a} = \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a}$$

Continuations of the Proofs

$$= \frac{f(x)-f(a)}{x-a} g(x) + f(a) \frac{g(x)-g(a)}{x-a} \xrightarrow[x\to a]{} f'(a)g(a) + f(a)g'(a)$$

The quotient rule:

To differentiate $\frac{f(x)}{g(x)}$, use product rule for $f(x) \cdot \frac{1}{g(x)}$: $f(x)$ & $\frac{1}{g(x)}$ are diff'ble at a,

$$\left(f(x)\cdot\frac{1}{g(x)}\right)'\Bigg|_{x=a} = f'(a)\frac{1}{g(a)} + f(a)\cdot\left(\frac{1}{g(x)}\right)'\Bigg|_{x=a}$$

$$= f'(a)/g(a) + f(a)\left(-\frac{g'(a)}{(g(a))^2}\right)$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2} \qquad \square$$

The Derivative of a Composite Function

Remark. A function f is differentiable at a ~~if~~ iff there is a number L such that the quotient

$$\lim_{h\to 0}\frac{f(a+h)-f(a)-Lh}{h}=0.$$

In this case $L=f'(a)$. Set $V(h):=\frac{f(a+h)-f(a)-Lh}{h}$. Then $V(h)\to 0$ and $f(a+h)=f(a)+Lh+V(h)h$.

$f'(a)$ exists $\Rightarrow \lim\limits_{h\to 0}\frac{f(a+h)-f(a)-f'(a)h}{h}$

$=\lim\limits_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)-f'(a)=f'(a)-f'(a)=0.$

If for some $L\in\mathbb{R}$, $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)-Lh}{h}=0$,

then $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim\limits_{h\to 0}\frac{f(a+h)-f(a)-Lh+Lh}{h}$

$=\lim\limits_{h\to 0}\frac{f(a+h)-f(a)-Lh}{h}+\lim\limits_{h\to 0}L=0+L=L$

The Proof of the Chain Rule

Theorem (Chain Rule)

Let I and J be open intervals. If $f : I \to J$ is differentiable at $a \in I$ and $g : J \to \mathbb{R}$ is differentiable at $f(a) \in J$, then the composition $(g \circ f)(x) := g(f(x))$ is differentiable at $a \in I$ and

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proof. $h := x - a$, $x = a + h$, $H = f(x) - f(a) \in \mathbb{R}$

$f'(a)$ exists $\Rightarrow f(x) - f(a) = f'(a)h + V(h)h$ where $V(h) \to 0$ as $h \to 0$.

$g'(f(a))$ exists $\Rightarrow$ $\forall H \in \mathbb{R}$, s.t. $f(a)+H \in J$, $g(f(a)+H) = g(f(a)) + g'(f(a))H + W(H)H$ for some $W(H) \to 0$ as $H \to 0$

Then $g(f(x)) = g(f(a)+H) = g(f(a)) + g'(f(a))(f'(a)h + V(h)h)$

The Proof of the Chain Rule, Continued

$$+ W(H)\left(f'(a)h + V(h)h\right)$$

$$= g(f(a)) + g'(f(a))f'(a)h$$

$$+ h\left(g'(f(a))V(h) + W(H)f'(a) + V(h)\right).$$

Since $g'(f(a))V(h) + W(H)f'(a) + V(h) \to 0$ as $h \to 0$, b/c $V(h) \to 0$, & and

$W(H) = W(f(a+h) - f(a)) \to 0$, because $f(a+h) - f(a) \to 0$ as f is conts at a and $W(H) \to 0$ as $H \to 0$. $\square$