Math 5615 Honors: Rearrangements and Riemann's theorem
Limits of Functions

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Rearrangements

Definition

Let $\sum_{k=1}^{\infty} a_k$ be a given series. Let $\{p_k\}$ be a sequence in which every positive integer occurs exactly once, that is, $p : \mathbb{N} \to \mathbb{N}$, $k \mapsto p_k$, is bijective. Then $\{p_k\}$ is called a *permutation* of $\mathbb{N}$ and the series $\sum_{k=1}^{\infty} a_{p_k}$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$.

Example

$$\sum_{k=1}^{\infty} (-1)^{k-1} = 1 - 1 + 1 - 1 + 1 - \dots$$

diverges but not to $\pm\infty$

Rearrange: $1 + 1 - 1 + \underbrace{1 + 1 + 1}_{3} - 1 + \underbrace{1 + 1 + 1 + 1}_{4} - 1 + \dots$

S_n's: 1, 2, 1, 2, 3, 4, 3, 4, 5, 6, 7, 6, 7, 8, 9, ...

diverges to ∞

Rearrangement Example

Example

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

via a sequence of illegal rearrangments, and subtractions

such as $1+2+3+4+5+\ldots$

$-(0+1+2+3+4+\ldots)$

$= 1+1+1+1+1+\ldots$

See a video on youtube

There is a rigorous sense in which $\sum_{k=1}^{\infty} k = -\frac{1}{12}$: $\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}$; $\zeta(-1) = \text{“}\sum_{k\geq 1} k\text{”} = -\frac{1}{12}$

(zeta)

By analytic continuation

Rearrangement of Absolutely Convergent Series

Theorem

If $\sum_{k=1}^{\infty} a_k$ converges absolutely and $S = \sum_{k=1}^{\infty} a_k$, and if $\sum_{k=1}^{\infty} a_{p_k}$ is any rearrangement of $\sum_{k=1}^{\infty} a_k$, then we also have $S = \sum_{k=1}^{\infty} a_{p_k}$.

Proof. Cauchy criterion for $\sum_{k=1}^{\infty} |a_k|$:

$\forall \varepsilon > 0 \ \exists N : \forall m > n \geq N$ $\sum_{k=n+1}^{m} |a_k| < \varepsilon/2$. Take $M \geq N$ such that $\{1, 2, \dots, N\} \subset \{p_1, p_2, \dots, p_M\}$. Then $\sum_{k=1}^{M} a_k$ and $\sum_{k=1}^{M} a_{p_k}$ will include $a_1, a_2, \dots, a_N$ and so will $\sum_{k=1}^{K} a_k$ and $\sum_{k=1}^{K} a_{p_k}$ $\forall K \geq M$.

Proof, Continued

Then $\forall K \geq M$ $\left|\sum_{k=1}^{K} a_k - \sum_{k=1}^{K} a_{p_k}\right| = \left|\sum_{k=N+1}^{K} a_k - \sum_{\substack{k=1 \\ \text{skipping} \\ a_1,\dots,a_N}}^{K} a_{p_k}\right|$

$\underset{\substack{\text{triangle} \\ \text{inequality}}}{\leq} \sum_{k=N+1}^{K} |a_k| + \sum_{\substack{k=1 \\ \text{skipping} \\ a_1,\dots,a_N}}^{K} |a_{p_k}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ by the choice of N

in the Cauchy criterion for $\sum_{k=1}^{\infty} |a_k|$.

Thus, $\forall \varepsilon > 0$ $\exists M \in \mathbb{N} : \forall K \geq M$
$\left|\sum_{k=1}^{K} a_k - \sum_{k=1}^{K} a_{p_k}\right| < \varepsilon.$ $\Rightarrow \lim_{K \to \infty} \left(\sum_{k=1}^{K} a_k - \sum_{k=1}^{K} a_{p_k}\right)$
exists and equals 0.

Proof, Concluded

But $\sum_{k=1}^{K} a_k$ has a limit, which is $= S$

sum of sequences $\Rightarrow$ $\sum_{k=1}^{K} a_{p_k}$ converges with limit S.

$\square$

Riemann's Theorem

Theorem

Suppose $\sum_{k=1}^{\infty} a_k$ is conditionally convergent. Given any real number S, there is a rearrangement $\sum_{k=1}^{\infty} a_{p_k}$ that converges to S.

Given a series $\sum_{k=1}^{\infty} a_k$, let

$$a_k^+ := \max\{a_k, 0\} \text{ and } a_k^- := \max\{-a_k, 0\}.$$

Then $a_k^+ = a_k$ if $a_k > 0$ and $a_k^+ = 0$ otherwise; $a_k^- = |a_k|$ if $a_k < 0$ and $a_k^- = 0$ otherwise.

Proposition

If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then the series $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ are both convergent. If $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then the series $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ are both divergent.

Proof of Proposition

(1) $\sum a_k$ converges absol. $\overset{\text{def}}{\iff} \sum_{k=1}^{\infty} |a_k|$ converges

$0 \le a_k^+ \le |a_k|$ Comparison $\Longrightarrow$ Test $\left\{ \begin{array}{l} \sum a_k^+ \\ \sum a_k^- \end{array} \right\}$ converge.

$0 \le a_k^- \le |a_k|$

(2) $\sum |a_k|$ diverges, $\sum a_k$ converges,

$|a_k| = a_k^+ + a_k^- \Rightarrow \sum_{k \ge 1} (a_k^+ + a_k^-)$ diverges

~~Both~~ $\sum a_k^+$ and $\sum a_k^-$ cannot ~~both~~ converge

Suppose $\underbrace{\sum a_k^+}_{\text{WLOG}} = S$ (converges), $\sum a_k^-$ diverges.

S_n, S_n^+, S_n^- : $S_n = S_n^+ - S_n^-$ $\forall n$, because
(partial sums) $a_k = a_k^+ - a_k^-$.

Proof of [redacted]

<u>Proof of Prop. Continued:</u> $\{S_n^-\}$ diverges to ∞ $\Rightarrow$

$\forall M>0 \ \exists N>0 : \forall n \geq N \quad S_n^- > M+S.$

Also, $S_n^+ \leq S$ ($\{S_n^+\}$ is increasing and $S_n^+ \to S$)

$$\Rightarrow S_n = S_n^+ - S_n^- < S - (M+S) = -M$$

$$\Rightarrow S_n \to -\infty$$ Contradiction.

Prop. proven.

Riemann's Theorem to be proven on Wednesday.