Math 5615 Honors: More Uses of the Derivative

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Darboux's Theorem

If a function f has a derivative f' on an open interval, f' does not have to be continuous, but it has shares one property with continuous functions: the intermediate value theorem.

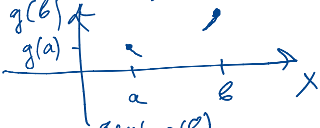
Theorem (Darboux)

Let I be an open interval of the real line, and suppose $f : I \to \mathbb{R}$ is a differentiable function. Then f' has the following intermediate value property on I: If $a, b \in I$ with $a < b$ and $f'(a) \neq f'(b)$, then for any number m between $f'(a)$ and $f'(b)$, there is a point $c \in (a, b) \subset I$ such that $f'(c) = m$.

Proof. Suppose $f'(a) < m < f'(b)$. Then $g(x) := f(x) - m(x-a)$, $x \in I$, is differentiable, and $g'(x) = f'(x) - m$. Note $g'(a) = f'(a) - m < 0$ and $g'(b) = f'(b) - m > 0$. $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} < 0 \Rightarrow$

Proof of Darboux's Theorem, Continued

$\Rightarrow$ For small enough $x-a>0$, we have $\frac{g(x)-g(a)}{x-a}<0$ and thereby $g(x)-g(a)<0$.



Likewise, since $g'(b)=\lim_{x\to b}\frac{g(x)-g(b)}{x-b}>0$, for x close enough to b and $x-b<0$ we have $\frac{g(x)-g(b)}{x-b}>0$ and thereby $g(x)-g(b)<0$.

If $f'(a)>f'(b)$

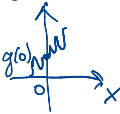
Thus, the abs. minimum of $g(x)$ on $[a,b]$ is $<g(a)$ and $g(b)$. Therefore it's attained (recall g is conts on compact $[a,b]$) at $c\neq a,b$; $c\in[a,b]$. Thus, c is a pt. of local minimum of g on (a,b) and $g'(c)=0$.
$\Rightarrow f'(c)=m$. If $f'(a)>m>f'(b)$, then similarly show that $g(x)$ has a local max. at $c\in(a,b)$.

Examples

1. $$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$. But $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ for $x \neq 0$ does not have a limit as $x \to 0$, and therefore $f'(x)$ is not continuous at 0. Thus, f' could be not continuous, but it still satisfies the intermediate value theorem. (Recall that f is differentiable (and $f'(0)=0$) on $\mathbb{R}$)

2. $g(x) = f(x) + x/2$ has $g'(0) = 1/2 > 0$ but not increasing on any interval about 0. Nevertheless, $g(x) < g(0)$ for all small enough $x < 0$ and $g(0) < g(x)$ for all small enough $x > 0$.



3. Darboux's theorem implies: no jump discontinuities for $f'(x)$.

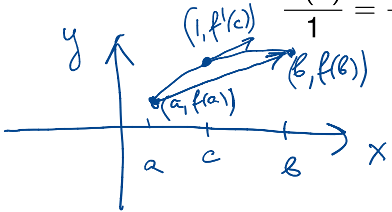
[redacted] Mean Value Theorem (MVT)

MVT: If f is conts on $[a, b]$ and diffble on (a, b), then at some $c \in (a, b)$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The tangent vector to the curve with parametric equations $x = t$, $y = f(t)$ (the graph of $y = f(x)$) at point c is $(1, f'(c))$ is parallel to the vector $(b - a, f(b) - f(a))$ from $(a, f(a))$ to $(b, f(b))$, because the slopes are equal:

$$\frac{f'(c)}{1} = \frac{f(b) - f(a)}{b - a}.$$



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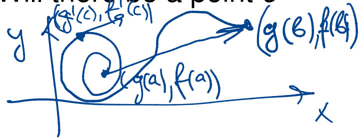
Cauchy's Mean Value Theorem (MVT)

A more general plane curve would be given by parametric equations $x = g(t)$, $y = f(t)$. The tanget vector at point $t = c$ is $(g'(c), f'(c))$, whereas the vector from $(g(a), f(a))$ to $(g(b), f(b))$ is $(g(b) - g(a), f(b) - f(a))$. Will there be a point c at which they are parallel:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}?$$



Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on the open interval (a, b) and such that $f'(x)$ and $g'(x)$ are not both equal to 0 at any $x \in (a, b)$ and $g(b) \neq g(a)$. Then there is a point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof of Cauchy's MVT

Proof. As in the proof of MVT, consider

MVT: $h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$

$$h(x) := f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Then $h(a) = h(b) = 0$, conts on $[a, b]$, diffble on (a, b) and Rolle's theorem applies: $\exists\ c \in (a,b)$:

$$0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c)$$

But $g'(c) \neq 0$, because otherwise $f'(c) = 0$ also. Thus, can divide by $g'(c)$ and get

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}. \quad \square$$