Math 5615 Honors: L'Hôpital's Rule [redacted]
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L’Hôpital’s Rule

$$\lim_{x\to 0} \frac{x}{e^x-1} = \lim_{x\to 0} \frac{x'}{(e^x-1)'} = \lim_{x\to 0} \frac{1}{e^x} = \frac{1}{e^0} = 1$$

Theorem (Rule for 0/0 Forms)

Suppose f and g are defined and continuous on an open interval I containing ~~the~~ a point a, f and g are differentiable on $I \setminus \{a\}$, $f(a) = g(a) = 0$, $g'(x) \neq 0$ for $x \in I \setminus \{a\}$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof

If $g(x) \neq 0$ for $\forall x \in I \setminus \{a\}$, then Cauchy's MVT gives

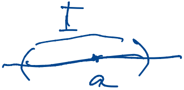
$$\frac{f(x) - \cancel{f(a)}}{g(x) - \cancel{g(a)}} = \frac{f'(c)}{g'(c)}$$ for some c between a & x.

(If $g(x) = 0$ for some $x \in I \setminus \{a\}$, then MVT gives $0 = g(x) - g(a) = g'(c_1)(x-a)$ for some c_1 between a & x, implying $g'(c_1) = 0$, which contradicts $g'(x) \neq 0$ $\forall x \in I \setminus \{a\}$.)

As $x \to a$, $c \to a \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)}$. $\square$

More detail: $\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon$ for $\forall c: 0 < |c-a| < \delta$ and therefore $\forall x: 0 < |x-a| < \delta$

L'Hôpital's Rule for ∞/∞ Forms



Suppose $I \subset \mathbb{R}$ open interval, $a \in I$.

Theorem (Rule for ∞/∞ Forms)

Suppose f and g are defined and ~~continuous on an open interval I containing the point a, f and g are~~ differentiable on $I \setminus \{a\}$, $\lim_{x\to a} f(x) \underset{=\pm\infty}{\cancel{=}} \lim_{x\to a} g(x) = \pm\infty$, $g'(x) \neq 0$ for $x \in I \setminus \{a\}$, and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof

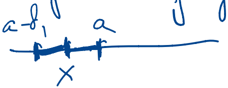
Let $\delta > 0$: $(a-\delta, a+\delta) \subset I$, $L := \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Then $\forall \varepsilon > 0 \ \exists \delta_1 : 0 < \delta_1 < \delta$ s.t.

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon \text{ for all } x : 0 < |x-a| < \delta_1.$$

Cauchy's MVT applies to f, g on $[a-\delta_1, x]$

for any given x : $a-\delta_1 < x < a$: $\exists c \in (a-\delta_1, x)$



$$\frac{f(x) - f(a-\delta_1)}{g(x) - g(a-\delta_1)} = \frac{f'(c)}{g'(c)}$$

(If $x \in (a, a+\delta_1)$

$\exists c \in (x, a+\delta_1)$:

$$\frac{f(x) - f(a+\delta_1)}{g(x) - g(a+\delta_1)} = \frac{f'(c)}{g'(c)}.$$

This case is dealt with similarly to below.)

Proof

Note:
$g(x)-g(a-\delta_1)\neq 0 \ \forall x\in I\setminus\{a\}$, otherwise

by MVT $g'(d)=0$ for some d betw. $a-\delta_1$ & x

thus, $\left|\dfrac{f(x)-f(a-\delta_1)}{g(x)-g(a-\delta_1)}-L\right|<\varepsilon \ \forall x$

$0<|x-a|<\delta_1$.

$$\varepsilon>\left|\frac{f(x)-f(a-\delta_1)}{g(x)-g(a-\delta_1)}-L\right|=\left|\frac{f(x)}{g(x)}\cdot\underbrace{\frac{1-\frac{f(a-\delta_1)}{f(x)}}{1-\frac{g(a-\delta_1)}{g(x)}}}_{h(x)}-L\right|$$

but $\lim_{x\to a}\left(1-\dfrac{f(a-\delta_1)}{f(x)}\right)=1=\lim_{x\to a}\left(1-\dfrac{g(a-\delta_1)}{g(x)}\right)$ and

$$\left|\frac{f(x)}{g(x)}-L\right|\leq\left|\frac{f(x)}{g(x)}-\frac{f(x)}{g(x)}h(x)\right|+\left|\frac{f(x)}{g(x)}h(x)-L\right|$$

Complete the argument here $<\varepsilon$ for x suff. close to a.