Math 5615 Honors: Taylor's Theorem and Higher-Derivative Test for Relative Extrema

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Higher Derivatives

Suppose $f : I \to \mathbb{R}$ has a derivative f' on an open interval I and f' is also differentiable on I. This derivative $(f')'$ is denoted by f'' and called the *second derivative of f*. If f'' happens to be differentiable on I, we call its derivative $f^{(3)} = f''' := (f'')'$ the *third derivative of function f*. **If this process can be iterated**, we obtain the *nth derivative*

$$f^{(n)} := (f^{(n-1)})'$$

of f. By convention, $f^{(0)} := f$.

Higher Mean Value Theorem

Lemma (Higher Mean Value Theorem)

Let I be an open interval and let n be a nonnegative integer. Suppose that $f : I \to \mathbb{R}$ has $n+1$ derivatives $f', f'', \ldots, f^{(n+1)}$ on I, and that at some point a in I,

$$f^{(k)}(a) = 0 \qquad \text{for } 0 \le k \le n.$$

Then for each $x \neq a$ in I, there is a point c in I between a and x such that

$$f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Cf. to MVT ($n = 0$): (if $f(a) = 0$, f' exists on I)

$$\frac{f(x) - f(a)}{x - a} = \frac{f(x)}{x - a} = f'(c). \Rightarrow f(x) = \frac{f'(c)}{1!}(x-a)$$

Proof

Set $g(x) := \frac{1}{(n+1)!}(x-a)^{n+1}$. Then

$$g^{(k)}(a) = 0 \quad \text{for all } k; \; 0 \le k \le n$$

$\left(g'(x) = \frac{1}{n!}(x-a)^n, \text{ etc.}, \right.$
$\left. g^{(n)}(x) = x-a, \; g^{(n+1)}(x) \equiv 1 \right)$

Note if $x \neq a, x \in I, g(x) \neq 0, g'(x) \neq 0, \ldots, g^{(n)}(x) \neq 0$. Apply Cauchy's MVT to $f(x)$ & $g(x)$ on (a, x) (Assume WLOG that $x > a$):

$$\frac{f(x) - \cancel{f(a)}}{g(x) - \cancel{g(a)}} = \frac{f'(c_1)}{g'(c_1)} \text{ for some } c_1 \in (a, x).$$

Proof

On $[a, c_1]$: $a < c_1 < x$, by Cauchy's MVT applied to $f'(x), g'(x)$: $\exists\, c_2 \in (a, c_1)$ s.t.

$$\frac{f'(c_1)}{g'(c_1)} = \frac{f'(c_1) - f'(a)}{g'(c_1) - g'(a)} = \frac{f''(c_2)}{g''(c_2)}$$

$a < c_2 < c_1 < x$

Continue:

$$\frac{f(x)}{g(x)} = \frac{f'(c_1)}{g'(c_1)} = \frac{f''(c_2)}{g''(c_2)} = \dots = \frac{f^{(n)}(c_n)}{g^{(n)}(c_n)} = \frac{f^{(n+1)}(c)}{g^{(n+1)}(c)}$$

for some pts $a < c < c_n < c_{n-1} < \dots < c_2 < c_1$

Note $g^{(n+1)}(c) = 1$. Thus, $f(x)/g(x) = f^{(n+1)}(c)$! □

Taylor's Theorem with Lagrange Remainder

We define the *Taylor polynomial of degree n for f at a point a* by

$$P_n(x) := f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Notice that the derivatives of P_n and f at a are equal through order n: $P_n'(x) = f'(a) + \frac{f''(a)}{1!}(x-a) + \ldots + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$, etc.

Theorem (Taylor)

Let I be an open interval containing a and let n be a nonnegative integer. If $f : I \to \mathbb{R}$ has $n+1$ derivatives on I, then for any $x \neq a$ in I there is a point c in I between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Proof of Taylor's Theorem

Let $R_n(x) := f(x) - P_n(x)$.

Apply Lemma to $R_n(x)$. Indeed, $R_n(x)$ is $(n+1)$ times diff'ble on I, $R_n^{(k)}(a) = f^{(n)}(a) - P_n^{(n)}(a)$

$= 0 \ \forall\, 0 \le k \le n$

By Lemma $\forall x \in I\ \exists\, c$ in I, betw. a & x:

$$f(x) - P_n(x) = R_n(x) = \frac{R_n^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Note $R_n^{(n+1)}(c) = f^{(n+1)}(c) - P_n^{(n+1)}(c)$

$= f^{(n+1)}(c)$. Thus, $f(x) = P_n(x) + \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$. $\square$

The $n = 1$ Case

Compare

$n=0$ $f(x) = f(a) + \frac{f'(c)}{1!}(x-a)$

$\Leftrightarrow \frac{f(x)-f(a)}{x-a} = f'(c)$, i.e., MVT

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2!}(x-a)^2$$

under the hypothesis that $f''(x)$ exists on I to

$$f(x) = f(a) + L(x-a) + V(x)(x-a),$$

where $L = f'(a)$ and $\lim_{x\to a} V(x) = 0$ (from an equivalent definition of $f'(a)$ – under hypothesis that $f'(a)$ exists).

Under the hypothesis $f''(x)$ exists on I,

$V(x) = \frac{f''(c)}{2!}(x-a)$ with c between a & x.