Math 5615 Honors: Proof Riemann's Theorem on Rearrangements Limits of Functions [redacted]

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Riemann's Theorem

Theorem

Suppose $\sum_{k=1}^{\infty} a_k$ is conditionally convergent. Given any real number S, there is a rearrangement $\sum_{k=1}^{\infty} a_{p_k}$ that converges to S.

Given a series $\sum_{k=1}^{\infty} a_k$, let

$$a_k^+ := \max\{a_k, 0\} \text{ and } a_k^- := \max\{-a_k, 0\}.$$

Then $a_k^+ = a_k$ if $a_k > 0$ and $a_k^+ = 0$ otherwise; $a_k^- = |a_k|$ if $a_k < 0$ and $a_k^- = 0$ otherwise.

Proposition (Proved last time)

If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then the series $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ are both convergent. If $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, then the series $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ are both divergent.

Proof of Riemann's Theorem

Know $\sum_k a_k^+$, $\sum_k a_k^-$ diverge, $\sum_k a_k$ converges $\Rightarrow a_k \to 0$ as $k \to \infty$. Suppose $S \geq 0$.

Construct a rearrangement of $\sum a_k$ as per algorithm:

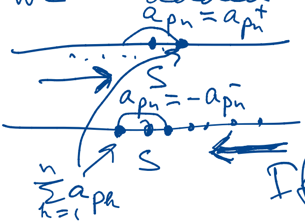
1. Add a_k^+'s from $\sum a_k$ (in their original order) up to the first a_k^+ term so that we exceed S (possible because $\sum a_k^+$ diverges to ∞)
2. Add negative terms $-a_k^-$ of $\sum a_k$ (in their original order) up to the first a_k^- term so that the resulting sum $< S$. (possible b/c $\sum a_k^-$ diverges to ∞).
3. Repeat steps 1 & 2. Never terminates b/c $\sum a_k^-$, $\sum a_k^+$ diverge.

Get a rearrangement $\sum_{k=1}^{\infty} a_{p_k}$.

Proof of Riemann's Theorem, Continued

Given $\varepsilon > 0$, $\exists N$: $\forall k \geq N$ $|a_k| < \varepsilon$. Choose K: $a_1, \dots, a_N$ are among $a_{p_1}, a_{p_2}, \dots, a_{p_K}$. Thus, if $k \geq K$, then $|a_{p_k}| < \varepsilon$.

<u>Claim</u>: $\forall n \geq K$ $\left|S - \sum_{k=1}^{n} a_{p_k}\right| < \varepsilon$. Otherwise, if $\left|S - \sum_{k=1}^{n} a_{p_k}\right| \geq \varepsilon$ for some $n \geq K$, then we've added too many terms of the same sign in the algorithm.



Thus, $\sum_{k=1}^{\infty} a_{p_k}$ converges to S.

If $S < 0$, then start with step 2.

Example

Take the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots,$$

which converges and has sum $\log 2$ (the Taylor series of ~~$\log x$~~ $\log(1+x)$).
The rearrangement

$$\left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \frac{1}{6} + \dots$$

converges to $\frac{3}{2}\log 2$, being $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k} + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$.
The parentheses reflect the algorithm used in the proof of Riemann's theorem to achieve $S = \frac{3}{2}\log 2$.

$\left(\frac{1}{5}+\frac{1}{7}\right) > \frac{1}{4} > \frac{1}{9}+\frac{1}{11}$, etc.

Example, Continued

$$
\begin{array}{l}
\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \ldots \\
+\ 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + \ldots \\
\hline
\ 1 + \cancel{0} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \cancel{0} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cancel{0} + \frac{1}{11} - \ldots
\end{array}
$$

$$
= \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \ldots
$$

Limit of a Function

Definition

Let X and Y be metric spaces (important case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$), $D \subset X$ and $f : D \to Y$. Let a be a cluster point of D, and let $L \in \cancel{X}\, Y$. We say that f *has limit* L *as* x *approaches* a, and write $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

($f(x) \to L$ as $x \to a$)

$$x \in D \text{ and } 0 < d(x,a) < \delta \Rightarrow d(f(x), L) < \varepsilon.$$

Theorem

If, under the assumption of the definition above, $\lim_{x\to a} f(x) = L_1$ *and* $\lim_{x\to a} f(x) = L_2$*, then* $L_1 = L_2$*.*

Proof.

$\forall \varepsilon > 0$ $d(L_1, L_2) \le d(L_1, f(x)) + d(L_2, f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

if we choose x close enough to a:

$0 < d(x,a) < \delta$. Then we have

and $d(L_1, L_2) = 0$. $\square$