Math 5615 Honors: Properties of Continuous Functions

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Correction: Continuity

Today: X and Y are metric spaces.

Definition

Let $D \subset X$ and $f : D \to Y$. If $a \in D$, then f is *continuous at a* if

$$\forall \varepsilon > 0, \exists \delta > 0 : \left\{ \overset{x \in D}{d(x, a) < \delta} \right\} \Rightarrow d(f(x), f(a)) < \varepsilon.$$

(or, equivalently, $f(D \cap B_\delta(a)) \subset B_\varepsilon(f(a))$)

Theorem

Let $D \subset X$ and $f : D \to Y$. If $a \in D$ and a is a cluster point of D, then f is continuous at a iff

$$\lim_{x \to a} f(x) = f(a).$$

Proof. By definition,

$$\lim_{x \to a} f(x) = f(a)$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : \left\{ \overset{x \in D}{0 < d(x, a) < \delta} \right\} \Rightarrow d(f(x), f(a)) < \varepsilon.$$

Continuation of the Proof

Since $d(f(a), f(a)) = 0$, the condition $0 < d(x, a)$ may be removed without affecting the statement in the previous line. This makes it equivalent to the above definition of continuity at a.

Continuity on a Domain

Definition

We say that a function $f : D \subset X \to Y$ is *continuous on D* if it is continuous at each $a \in D$.

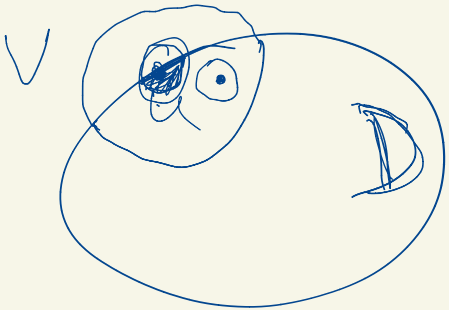
Definition

$U \subset D \subset X$ is *open relative to D* if there exists an open set $V \subset X$ such that $V \cap D = U$. Equivalently, U is *open relative to D* if U is open in D as a metric space, with the metric space structure (distance function) coming from that of X.

Theorem

A function $f : D \to Y$ is continuous on D if and only if the inverse image $f^{-1}(V) := \{x \in D \mid f(x) \in V\}$ of every open set $V \subset Y$ is open relative to D. If the domain D is an open set in X, then f is continuous on D if and only if the inverse image $f^{-1}(V)$ of every open set $V \subset Y$ is open.

Picture to illustrate open relative to D sets:



X

$U = V \cap D$

V open in X

U rel. open in D

<u>Remark:</u>

If D is open, then $U \subset D$ open relative to D iff U is open in X. Indeed,

($\Rightarrow$) $\exists\, V$ open in X: $U = V \cap D \Rightarrow U$ open in X

($\Leftarrow$) If U is open in X, $U \subset D$, take $V = U \subset X$ open in X. Then $U = V \cap D$.

Proof

$\Rightarrow$ Suppose f is conts on D, $V \subset Y$ open.

If $f^{-1}(V) = \emptyset$, then it's open (and rel. open) for trivial reasons. Otherwise, $\forall a \in f^{-1}(V)$, $f(a) \in V$. Then $\exists\, \varepsilon > 0$: $B_\varepsilon(f(a)) \subset V$.

f conts at a $\Rightarrow$ $\exists\, \delta = \delta(a) > 0$: $f(D \cap B_\delta(a)) \subset B_\varepsilon(f(a)) \subset V$,

i.e., $D \cap B_\delta(a) \subset f^{-1}(V)$. (This implies a is an interior pt of $f^{-1}(V) \subset D$ viewed as a metric space)

$$U_1 := \bigcup_{a \in f^{-1}(V)} B_{\delta(a)}(a) \subset X$$

Claim: $U_1 \cap D = f^{-1}(V)$ and $U_1 \subset X$ open.

(See why on your own.)

Proof, Continued

$\Leftarrow$ Suppose $\forall$ open $V \subset Y$ $\exists$ open $U \subset X$ s.t. $f^{-1}(V) = U \cap D$. Let $a \in D$. Want to show f is conts at a. $\forall \varepsilon > 0$ $B_\varepsilon(f(a))$ is open and $a \in f^{-1}(B_\varepsilon(f(a)))$. Then $\exists$ open $U \subset X$:

$a \in f^{-1}(B_\varepsilon(f(a))) = U \cap D$. Take δ: $B_\delta(a) \subset U$.

Then $D \cap B_\delta(a) \subset D \cap U = f^{-1}(B_\varepsilon(f(a)))$

$\Rightarrow f(D \cap B_\delta(a)) \subset B_\varepsilon(f(a))$

and f is conts at a. $\square$

Corollary for closed sets

Corollary

A function $f : D \to Y$ is continuous on D if and only if the inverse image $f^{-1}(C) := \{x \in D \mid f(x) \in C\}$ of every closed set $C \subset Y$ is closed relative to D. If the domain D is a closed set in X, then f is continuous on D if and only if the inverse image $f^{-1}(C)$ of every closed set $C \subset Y$ is closed.

Proof. ~~Def~~ closed rel. to D means intersection of a closed subset in X with D,

C is closed $\Leftrightarrow$ $C^c = Y \setminus C$ is open

$$f^{-1}(C^c) \underset{\supseteq}{\overset{\subseteq}{=}} f^{-1}(C)^c \cap D = D \setminus f^{-1}(C)$$

(Exercise) $\square$

Theorem

If $f : D \subset X \to Y$ is continuous and D is a connected set in X, then $f(D)$ is a connected set in Y.

Proof. Suppose $f(D)$ is disconnected, i.e.,
$\exists$ rel. open U_1, V_1 in $f(D)$: $U_1 \neq \emptyset$, $V_1 \neq \emptyset$,
$U_1 \cap V_1 = \emptyset$, $U_1 \cup V_1 = f(D)$.
f conts $\Rightarrow f^{-1}(U_1)$, $f^{-1}(V_1)$ are rel. open in D. They are $\neq \emptyset$, disjoint and
$f^{-1}(U_1) \cup f^{-1}(V_1) = D$, b/c $U_1 \cup V_1 = f(D)$
Thus, D must be disconnected $\square$

Intermediate Value Theorem

Corollary

If $f : [a, b] \to \mathbb{R}$ is a continuous function, and c is any real number strictly between $f(a)$ and $f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

Proof. $f([a,b])$ is connected by Theorem

$\Rightarrow$ it's an interval $I = f([a,b])$

$f(a), f(b) \in I$. If c is strictly between $f(a)$ & $f(b)$, then $c \in I = f([a,b])$.

So, $f(x) = c$ for some $x \in [a,b]$.

Actually, $x \neq a$ or b, b/c $c \neq f(a)$ or $f(b)$! $\Rightarrow x \in (a,b)$. $\square$