Problem 5 on midterm:

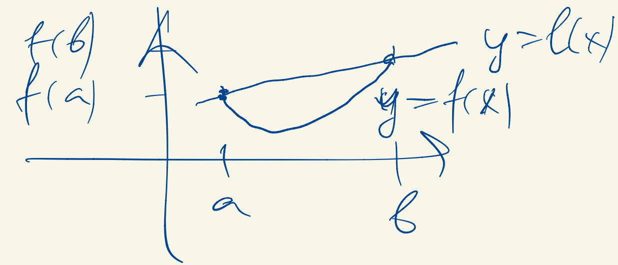
$f(x) : [a, b] \to \mathbb{R}$ conts and diff'ble on (a, b), $f'(x)$ is strictly increasing on (a, b)

Show $f(x)$ is convex on $[a, b]$, i.e.,

$\forall x \in (a, b)\ f(x) < \ell(x)$, where

$$\ell(x) = m(x-a) + f(a)$$

$$m = \frac{f(b) - f(a)}{b-a}$$



$g(x) = \ell(x) - f(x) > 0 \quad \forall x \in (a, b)$ — need to show

$$g(a) = 0, \quad g(b) = \frac{f(b) - f(a)}{b-a}(b-a) + f(a) - f(b) = 0$$

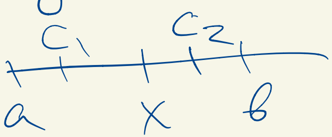
Suppose $\exists\, x \in (a,b) : g(x) \le 0$

Then apply MVT on $[a,x]$ and $[x,b]$;

$[a,x]$: $\exists\, c_1 \in (a,x) : g(x) - g(a) = g'(c_1)(x-a)$

$[x,b]$: $\exists\, c_2 \in (x,b) : g(b) - g(x) = g'(c_2)(b-x)$.

$g'(x) = m - f'(x)$ strictly decreasing



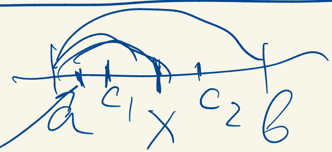
$c_1 < x < c_2 \Rightarrow g'(c_1) > g'(c_2)$

$0 \ge g(x) = g'(c_1)(x-a) \Rightarrow g'(c_1) \le 0$, b/c $x-a>0$

$0 \ge g(x) = g'(c_2)(x-b) \Rightarrow g'(c_2) \ge 0$, b/c $x-b<0$

$\Rightarrow g'(c_1) \le g'(c_2)$. This contradicts $g'(c_1) > g'(c_2)$.

Dead-end way: could be c_2



MVT on $[a,x] \rightsquigarrow c_1 \in (a,x)$

MVT on $[a,b] \rightsquigarrow c_2 \in (a,b)$ inconclusive as $c_1 \not< c_2$

4. **Common mistake:** $f: \mathbb{R} \to \mathbb{R}$

$$\frac{(f(x)-f(y))^2}{(x-y)^2} \leq |x-y| \quad \forall x \neq y \implies$$

$$\lim_{x \to y} \left(\frac{f(x)-f(y)}{x-y}\right)^2 \leq \lim_{x \to y} |x-y| \quad \text{true}$$

What's true is Thm: $g(x) \leq h(x)$ $\forall x$ and $\lim_{x\to y} g(x)$ and $\lim_{x\to y} h(x)$ exist $\Rightarrow \lim_{x\to y} g(x) \leq \lim_{x\to y} h(x)$

But it's not true if one of lims does not exist

Correct way: $0 \leq \left(\frac{f(x)-f(y)}{x-y}\right)^2 \leq |x-y|$ By squeeze Thm since $\lim_{x\to y}|x-y| = 0$, $\lim_{x\to y}\left(\frac{f(x)-f(y)}{x-y}\right)^2$ exists and $=0$.

Another warning:

$$p(x) \le g(x) \le h(x)$$

suppose $\lim_{x \to y}$ of h & p exist

but $\lim_{x \to y} p(x) = a$, $\lim_{x \to y} h(x) = b$,

$$a < b$$

$$\not\Rightarrow a \le \lim_{x \to y} g(x) \le b$$

b/c some fncts $g(x)$ satisfying the inequality will not have a limit

Only when they are squeezed "tight", as in the Squeeze theorem.

3. $x_0 \neq \frac{1}{2}$ Need to find $\varepsilon > 0$ s.t.

$\forall \delta > 0 \ \exists x : |x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon$

Know $f(x_0) \neq \frac{1}{2}$ Take $\varepsilon = |f(x_0) - \frac{1}{2}|$.

(1) If $x_0 \in \mathbb{Q}$, $f(x_0) = x_0$ $\varepsilon = |x_0 - \frac{1}{2}|$

Given a $\delta > 0$, Take an irrational # closer to x_0 than δ (by density of irrt'ls in $\mathbb{R}$) and closer to x_0 than ε. Then

$$|f(x) - f(x_0)| = |1 - x - x_0| = |1 - 2x_0 - (x - x_0)|$$

$$\geq ||1 - 2x_0| - |x - x_0|| = |2\varepsilon - |x - x_0|| = 2\varepsilon - |x - x_0| > \varepsilon.$$

reverse triangle inequality; $\varepsilon = |\frac{1}{2} - x_0|$; $|x - x_0| < \varepsilon$

(2) If $x_0 \notin \mathbb{Q}$, see posted solutions.