Math 5615H: Honors: Introduction to Analysis
Countable Sets
The Complete Ordered Field of Real Numbers

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Zoom Session Rules of Conduct

- Your **video must be always on**; I may be removing you after 10 seconds of video being off, but you will be able to reenter as soon as you find out that you have been removed.
- Your **audio may be on or off**.
- You are encouraged to answer my questions, ask questions and interrupt me. You may ask questions out loud or by via Zoom chat. If you do not want the whole class to see your question, you may choose the option of private chat message to me on Zoom. If I do not pay attention to Chat, please interrupt me and say: “Sasha, would you, please check your Chat window?”
- **Reward system**: You may get a maximum of 1 point per class, basically for any math related activity.

The Cardinality of $\mathbb{N} \times \mathbb{N}$, First Proof

Proposition

$\mathbb{N} \times \mathbb{N}$ is countable.

Proof.

I way: Cantor's diagonal process.

	1	2	3	4	...
1	1	3	6	10	
2	2	5	9		
3	4	8			
4	7				
...					...

not needed

$\mathbb{N} \to \mathbb{N} \times \mathbb{N}$

$f(n) = a_n$

$a_1 = (1,1)$, $a_2 = (2,1)$, $a_3 = (1,2)$, ... along the arrows

The Cardinality of $\mathbb{N} \times \mathbb{N}$, Second Proof

II way: Check that the function

$$h : \mathbb{N} \times \mathbb{N} \to \mathbb{N},$$
$$h(m, n) = 2^m 3^n,$$

is one-to-one. Thus, h establishes a bijection $h : \mathbb{N} \times \mathbb{N} \to \operatorname{im} h$, which is an infinite subset of $\mathbb{N}$. $\square$

Product of Two Countable Sets

Corollary

C_1, C_2 countable $\Rightarrow$ So is $C_1 \times C_2$.

Proof.

Given $h_1 : \mathbb{N} \xrightarrow{\sim} C_1$ and $h_2 : \mathbb{N} \xrightarrow{\sim} C_2$, define a bijection

$$H(m,n) := (h_1(m), h_2(n)) : \mathbb{N} \times \mathbb{N} \xrightarrow{\sim} C_1 \times C_2.$$

$\square$

The Rationals are Countable

Corollary

$\mathbb{Q}$ *is countable.*

Proof.

The function

$$\begin{aligned} \mathbb{Z} \times \mathbb{N} &\to \mathbb{Q}, \\ (m, n) &\mapsto m/n, \end{aligned}$$

from countable $\mathbb{Z} \times \mathbb{N}$ to $\mathbb{Q}$ is onto. $\square$

The Union of Countables

Proposition

$\bigcup_{n=1}^{\infty} C_n$ is countable, if C_n's are (They are allowed to intersect).

Proof.

See text or just check that the map

$$\mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} C_n,$$
$$(n, m) \mapsto h_n(m),$$

is onto. Here $h_n : \mathbb{N} \xrightarrow{\sim} C_n$ are given bijections. $\square$

Definition of a Field

A *field* is a set F with $A : F \times F \to F$ and $M : F \times F \to F$ satisfying the axioms:

$x+y := A(x,y)$

$xy := M(x,y)$

addition

Comm	$x + y = y + x \ \forall x, y \in F$;
Assoc	$(x + y) + z = x + (y + z)$;
Zero	$\exists 0 \in F : x + 0 = 0 + x = x$, (unique);
Negative	$\forall x \in F, \exists y \in F : x + y = 0$, (unique $-x := y$);
Comm	$xy = yx \ \forall x, y \in F$;
Assoc	$(xy)z = x(yz)$;
Unit	$\exists 1 \in F, 1 \neq 0 : x \cdot 1 = 1 \cdot x = x$, (unique);
Inverse	$\forall x \neq 0 \in F, \exists y \in F : xy = 1$, (unique $x^{-1} := y$);
Distrib	$x(y + z) = xy + xz$.

mult.

$0_1 + 0_2 = 0_2$, $\Rightarrow$

$0_1 + 0_2 = 0_1$

$\cdot\ 0_1 = 0_2$

Examples of Fields

$$\mathbb{Q} = \{ m/n \mid m \in \mathbb{Z}, n \in \mathbb{N} \}$$

$\mathbb{R}$ (e.g., think of decimal fractions, like $\pi = 3.14159\ldots$), but we will assume it is a field for now!

$\mathbb{Z}_2 = \{0, 1\}$ with

$$0+0=0,\ 0+1=1,\ 1+1=0$$
$$0\cdot 0=0,\ 0\cdot 1=0,\ 1\cdot 1=1$$

$\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R}$ with

$$(a,b)+(c,d) = (a+c,\ b+d)$$
$$(a,b)\cdot(c,d) := (ac-bd,\ ad+bc)$$
$$a+bi := (a,b)$$

$$1 = (1,0)$$
$$i = (0,1)$$
$$i^2 = (-1,0) = -1$$

Ordered Fields

Let F be a field and $P \subset F$ a subset that satisfies the following conditions:

Closure If $x, y \in P$, then $x + y \in P$ and $xy \in P$.

Trichtmy For each $x \in F$, exactly one of the following is true: $x \in P$, or $x = 0$, or $-x \in P$.

Then P is called a *positive set*. An *ordered field* is a field F that contains a positive set P.
Define a partial order on F: $x < y$ if $y - x \in P$, $x \leq y$ if $x < y$ or $x = y$. It is a total order, in fact.

Examples

$\mathbb{Q}$ with $P = \mathbb{Q}^+ = \{m/n \mid m \in \mathbb{N}, n \in \mathbb{N}\}$
$\mathbb{R}$ with $P = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$
$\mathbb{C}$ and $\mathbb{Z}_2$ are not ordered fields.