Math 5615H: Honors: Introduction to Analysis
The square root of 2
Complex numbers

Sasha Voronov

University of Minnesota

September 21, 2020

The existence of the square root $\sqrt{2}$

Theorem

There exists a unique positive real number r such that $r^2 = 2$.

Proof. Take $A := \{s \in \mathbb{R} \mid s \geq 0 \text{ and } s^2 \leq 2\}$ and $r = \sup A$...

exists, b/c A bdd above e.g., by 2 (if not, $\exists\ s \in A : s > 2 > 0$, $s^2 > 4 > 2$)

Want: $r^2 = 2$

$r^2 - 2 > 0$, < 0, or $= 0$

Idea: exclude $r^2 < 2$ and $r^2 > 2$.

(1) Suppose $r^2 < 2$. Let $\delta = 2 - r^2 > 0$.

We'll show: $\exists\, m \in \mathbb{N} : \left(r + \frac{1}{m}\right)^2 < 2$ contradicting the fact that r is an upper bd for A

Continuation of proving that $r^2 = 2$

Proof. $\left(r+\frac{1}{m}\right)^2 = r^2 + 2r\frac{1}{m} + \frac{1}{m^2}$

Archimedean property: $\exists\, m \in \mathbb{N}: m > \frac{2r}{\delta/2}$

and $m > \frac{1}{\delta/2}$

This is to make $\left(r+\frac{1}{m}\right)^2$ close enough to r^2: $\left(r+\frac{1}{m}\right)^2 < \underline{r^2+\delta} = 2$

Indeed, $\left(r+\frac{1}{m}\right)^2 = r^2 + 2r\frac{1}{m} + \frac{1}{m^2}$

$< r^2 + \frac{\delta}{2} + \frac{1}{m^2} \leq r^2 + \frac{\delta}{2} + \frac{1}{m} < r^2 + \frac{\delta}{2} + \frac{\delta}{2}$

$m^2 \geq m \;\; \forall m \in \mathbb{N}$

$m \geq 1$

Completion of showing that $r^2 = 2$

Proof. (2) Suppose $r^2 > 2$. Take $\delta = r^2 - 2 > 0$.

We'll show: $\exists\, m \in \mathbb{N} : \left(r - \frac{1}{m}\right)^2 > 2$

(This would imply that $r - \frac{1}{m}$ is an upper bd for A.

Otherwise, if "$r - \frac{1}{m}$ is not an upper bd", $\exists\, s \in A$: $0 < r - \frac{1}{m} < s$, then $\left(r - \frac{1}{m}\right)^2 < s^2 < 2$, which is a contradn)

Choose m: $m > \frac{1}{r}$ to have this

$$\left(r - \frac{1}{m}\right)^2 = r^2 - 2r\frac{1}{m} + \frac{1}{m^2} > r^2 - \left(2r\frac{1}{m} + \frac{1}{m^2}\right)$$

$\frac{1}{m^2} > -\frac{1}{m^2}$

unique; $0 < r_1 < r_2$, $r_1^2 < r_2^2$

$\exists\, m$: $m > \frac{2r}{\delta/2}$, $m > \frac{1}{\delta/2}$

Then $\left(r - \frac{1}{m}\right)^2 > r^2 - \left(\frac{\delta}{2} + \frac{\delta}{2}\right) = r^2 - \delta = 2$. $\square$

The Modulus

For $z = a + bi \in \mathbb{C}$, define $z = (a, b) = a + bi$, $i = (0, 1)$

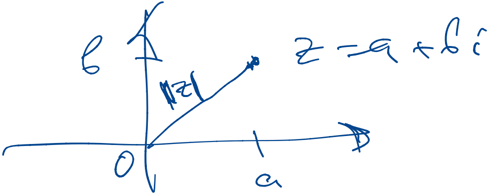
$$|z| = |a + bi| := \sqrt{a^2 + b^2},$$

$i^2 = -1$

$a = (a, 0)$
$bi = (0, b)$

called the *modulus* (or the *absolute value*) of z.

Figure:



b/c $\sqrt{a^2}$ is unique

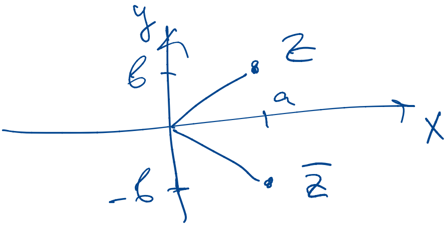
If $z \in \mathbb{R}$, that is, $z = a + 0i$, then $|z| = \sqrt{a^2} = |a|$. Indeed, $|a|^2 = a^2$, and a nonnegative square root is unique.

The Conjugate

For $z = a + bi \in \mathbb{C}$, define

$$\bar{z} = a - bi, \qquad \text{the } \textit{conjugate} \text{ of } z.$$

Figure:



$|\bar{z}| = |z|$

$$z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 + 0i = a^2 + b^2, \quad |z| = \sqrt{z\bar{z}}$$

Formula for z^{-1} (for $z = a + bi \neq 0$):

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{a^2 + b^2}.$$

Properties of the Modulus

Theorem

1. $|z| \geq 0$, *and* $|z| = 0$ *iff* $z = 0$;
2. $|zw| = |z||w|$;
3. $|z + w| \leq |z| + |w|$ *(the triangle inequality);*
4. $||z| - |w|| \leq |z + w|$ *(the reverse triangle inequality).*

Proof. (3) see text

(4) $||z|-|w|| \leq |z+w|$

$$\left.\begin{array}{l} |z|-|w| \leq |z+w| \\ \Updownarrow \\ |w|-|z| \leq |z+w| \end{array}\right\} \Rightarrow ||z|-|w|| \leq |z+w|$$

or

b/c $|x| = \begin{cases} x & \text{or} \\ -x \end{cases}$

$$|w| - |z| \le |z + w|$$

$$|z| \overset{(3)}{\le} |z + w| + \underbrace{|-w|}_{= |w|} \qquad \square$$