Math 5615H: Honors: Introduction to Analysis
Proof of Cauchy-Schwarz Inequality
Metric Spaces and Their Basic Topology

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September 25, 2020

Cauchy-Schwarz Inequality

Theorem

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *with its standard inner product* $(\mathbf{x}, \mathbf{y})$ *and norm* $|\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})}$, *we have*

$$|(\mathbf{x}, \mathbf{y})| \leq |\mathbf{x}| \cdot |\mathbf{y}|.$$

Equality holds iff the vectors $\mathbf{x}, \mathbf{y}$ *are collinear, that is, iff one is a scalar multiple of the other.*

Proof. Look at $(\mathbf{x} + t\mathbf{y}, \mathbf{x} + t\mathbf{y})$ as a real-valued function of $t \in \mathbb{R}$.

$(x+ty, x+ty) \geq 0 \quad \forall t \in \mathbb{R}$

$(x+ty, x+ty) = (x,x) + 2t(x,y) + t^2(y,y)$

Leave case $(y,y) = 0$ to the end and assume $(y,y) \neq 0$

Then $(y,y) > 0$ and

$\Rightarrow D = b^2 - 4ac \leq 0$

The Proof of Cauchy-Schwarz Inequality, Continued

$D \le 0$ rewrites as

$$4(x,y)^2 - 4(y,y)(x,x) \le 0$$

$$(x,y)^2 \le (x,x)(y,y)$$

$$\left(0 \le a \le b \Downarrow 0 \le \sqrt{a} \le \sqrt{b} \right)$$

$$|(x,y)| \le |x| \cdot |y|$$

Equality $\Leftrightarrow D = 0 \Leftrightarrow \exists\, t_0 \in \mathbb{R} : at_0^2 + bt_0 + c = 0$

i.e., $(x+t_0y, x+t_0y) = 0 \Leftrightarrow |x+t_0y| = 0$

$\Leftrightarrow x + t_0 y = 0 \Leftrightarrow x = -t_0 y$

Case $(y,y) = 0$ $\to t$ then $y = 0$. $|(x,0)| \le |x| \cdot |0|$ true

We always have ~~have~~ "=", $y = 0 \cdot x$, i.e., x, y are collinear. $\square$

Euclidean Distance

$$|x-y| \le |z-x| + |y-z|$$

In $\mathbb{R}^n$ *Euclidean distance (metric)*:

$$d(\mathbf{x},\mathbf{y}) := |\mathbf{x}-\mathbf{y}| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}.$$

Properties of the norm (e.g., $0 \le |\mathbf{x}+\mathbf{y}| \le |\mathbf{x}| + |\mathbf{y}|$, which may easily be derived from Cauchy-Schwarz) dictate properties of the distance in $\mathbb{R}^n$. We use them as axioms of a metric space.

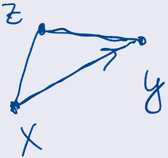
$$(|x|+|y|)^2 = (x,x) + 2|x|\cdot|y| + (y,y)$$
$$\ge (x,x) + 2|(x,y)| + (y,y)$$
$$\overset{C-S}{\ge} (x,x) + 2(x,y) + (y,y) = (x+y,x+y) \ge 0$$

Metric Spaces

Definition (Metric Space)

Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ denoted $d(x, y)$ is a *metric in* X if the following properties hold for $x, y, z \in X$:

1. $d(x, y) > 0$ if $x \neq y$;
2. $d(x, x) = 0$;
3. $d(x, y) = d(y, x)$;
4. $d(x, y) \leq d(x, z) + d(z, y)$.



A nonempty set X with a metric is called a *metric space*.

Examples

$\mathbb{R}^n$ with Euclidean distance, $\mathbb{C}$ with $d(z, w) := |z - w|$ (generalizes to $\mathbb{C}^n$). Math 5616H: $C[a, b]$, $R[a, b]$, $L^2[a, b]$.

Basic Topology in Metric Spaces

Definition

(X, d) a metric space. *Open ball*:

$$B_\delta(x) := \{y \in X \mid d(y, x) < \delta\}, \qquad \delta > 0$$

δ x

A subset $S \subset X$ is *bounded* if there is a number $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in S$.

S

If $S \subset X$, then $x \in X$ is called an *interior point* of S if there exists a number $\delta > 0$ such that the open ball $B_\delta(x) \subset S$.

x

A subset $S \subset X$ is called an *open set* if every element of S is an interior point of S. A subset $S \subset X$ is called a *closed set* if its complement, $X \setminus S$ is open.

0 $\mathbb{R}$ not open

$S = \{0\}$

$y \in B_\delta(x)$

Note: X and $\varnothing$ are open. Therefore, they are also closed. An open ball $B_\delta(x)$ is an open set.

x

Boundary, Cluster, and Isolated Points; Closure

Definition

If $S \subset X$, a point $x \in X$ is called a *boundary point* of S if every open ball about x contains at least one point of S and at least one point of $X \setminus S$. The *boundary* of S, denoted ∂S, is the set of boundary points of S.

If $S \subset X$, a point $x \in X$ is called a *cluster point* (*accumulation point*, or *limit point*) of S if every open ball about x contains infinitely many points of S (or, equivalently, one point of S apart from x). A point $x \in S$ which is not a cluster point of S is called an *isolated point* of S. The *closure* of S:

$$\overline{S} = S \cup \{\text{cluster points of } S\}.$$

$\mathbb{R}$ $\quad 0, \frac{1}{3}, \frac{1}{2}, 1, \frac{1}{n}$

0 is a cluster pt $\{\frac{1}{n}\}$

Characterization of Closed Sets

Theorem

For a subset $S \subset X$, TFAE:

1. *S is closed;* (i.e. $X \setminus S$ is open)
2. *S contains all its cluster points;*
3. $\overline{S} = S$.

Proof. (2) $\Leftrightarrow$ (3) clear

(1) $\Leftrightarrow$ (2) next time