Math 5615H: Honors: Introduction to Analysis
Basic Topology in Metric Spaces

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Characterization of Closed Sets

X is a metric space, $S \subset X$

Closure: $\overline{S} = S \cup \{\text{cluster points of } S\}$.

Theorem

For a subset $S \subset X$, TFAE:

1. *S is closed;*
2. *S contains all its cluster points;*
3. *$\overline{S} = S$.*

Proof.

(2) $\Leftrightarrow$ (3) obvious.

(1) $\Leftrightarrow$ (2): $\{S \text{ is closed }\} \overset{\text{def}}{\Leftrightarrow} \{X \setminus S \text{ is open}\}$

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$\Leftrightarrow \{x \in S \text{ iff } (x \in S \text{ or } \forall \delta > 0 \; \exists y \neq x : y \in B_\delta(x) \cap S)\}$

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Closure: $\overline{S} = S \cup \{\text{cluster points of } S\}$.

Theorem

For a subset $S \subset X$, TFAE:

$\neg(\exists \delta > 0 : B_\delta(x) \cap S = \emptyset)$
$= (\forall \delta > 0 \ B_\delta(x) \cap S \neq \emptyset)$

1. *S is closed;*
2. *S contains all its cluster points;*
3. *$\overline{S} = S$.*

Proof.

(2) $\Leftrightarrow$ (3) obvious. $\neg(x \notin S) = x \in S$

(1) $\Leftrightarrow$ (2): $\{S$ is closed $\} \Leftrightarrow \{X \setminus S$ is open$\} \Leftrightarrow \{x \notin S$ iff $\exists \delta > 0 : B_\delta(x) \cap S = \emptyset\} \Leftrightarrow \{x \in S$ iff $\forall \delta > 0\ B_\delta(x) \cap S \neq \emptyset\}$
$\Leftrightarrow \{x \in S$ iff $(x \in S$ or $\forall \delta > 0\ \exists y \neq x : y \in B_\delta(x) \cap S)\}$
$\Leftrightarrow \{x \in S$ iff $(x \in S$ or $x \in \overline{S})\} \Leftrightarrow \{\overline{S} = S\}$ $\square$

Double Closure

Corollary

$\overline{\overline{S}} = \overline{S}$, and hence $\overline{S}$ is a closed set.

Proof. Need to show $\overline{\overline{S}} \subset \overline{S}$, i.e., $\forall$ cluster pt of $\overline{S}$ is in $\overline{S}$.

sppse x cluster pt of $\overline{S}$

$\forall\, \delta > 0$ $B_\delta(x) \ni y \in \overline{S}$, $y \neq x$

$d(x,y) < \delta$, take $\delta' = \delta - d(x,y) > 0$

$B_{\delta'}(y)$. Since $y \in \overline{S}$, $\exists\, z$, $z \in B_{\delta'}(y) \cap S$. Thus $z \in B_{\delta'}(y) \subset B_\delta(x)$ Need $z \neq x$

Fine-tune δ' to be $\min(\delta - d(x,y), d(x,y))$. Then $B_{\delta'}(y) \not\ni x$.

Then x is a cluster pt of $\overline{S}$. $\Rightarrow x \in \overline{S}$. $\square$

Dense and Nowhere Dense Subsets

Definition

A subset $S \subset X$ is *dense* in X if $\overline{S} = X$. A set S is defined to be *nowhere dense* if $\overline{S}$ has no interior points.

Examples

$S=\mathbb{Q}\subset\mathbb{R}=X$, $\overline{\mathbb{Q}}=\mathbb{R}$

$\overline{(\mathbb{R}\setminus\mathbb{Q})}=\mathbb{R}$, $\mathbb{Z}\subset\mathbb{R}$, $\overline{\mathbb{Z}}=\mathbb{Z}$

$\overline{\mathbb{R}\setminus\mathbb{Z}}=\mathbb{R}$

n $n+1$

$\mathbb{Z}$ nowhere dense set in $\mathbb{R}$

$\overline{\mathbb{Z}}=\mathbb{Z}$ $n\in\mathbb{Z}$

Another example: Cantor set (later)

Union and intersection of open and closed sets

Theorem

In a metric space, the following statements are true:

1. *The union of any collection of open sets is open;*
2. *The intersection of any finite collection of open sets is open;*
3. *The intersection of any collection of closed sets is closed;*
4. *The union of any finite collection of closed sets is closed.*

Examples

(1) $\bigcup_{\alpha\in\Gamma} O_\alpha$, $O_\alpha \subset X$ open

$\forall x \in \bigcup_\alpha O_\alpha$, $x \in O_{\alpha_0}$ for some α_0

$\Rightarrow \exists\, \delta > 0$ $B_\delta(x) \subset O_{\alpha_0} \subset \bigcup_\alpha O_\alpha$

Proof of Theorem

(2) $\bigcap_{i=1}^{n} O_i$, $O_i \subset X$ open,

$\forall x \in \bigcap O_i$. $\forall i$ $\exists \delta_i > 0$: $B_{\delta_i}(x) \subset O_i$

$\delta = \min(\delta_1, \dots, \delta_n) > 0$, $B_\delta(x) \subset O_i \; \forall i$

$\Rightarrow B_\delta(x) \subset \bigcap_{i=1}^{n} O_i$.

(3) $\bigcap_{\alpha \in \Gamma} C_\alpha$, $C_\alpha \subset X$ closed, $C_\alpha^c = X \setminus C_\alpha$ open

$\left(\bigcap_{\alpha \in \Gamma} C_\alpha\right)^c = \bigcup_{\alpha \in \Gamma} \left(C_\alpha^c\right)$ open by (1)

(4) similar $\square$

Open Covers and Compact Sets

Definitions. A collection $\{O_\gamma \mid \gamma \in \Gamma\}$ of open subsets of a m. sp. X is an <u>open cover</u> of a subset $S \subset X$, if $\bigcup_\gamma O_\gamma \supset S$.
An open cover is <u>finite</u>, if $|\Gamma| < \infty$.
$K \subset X$ is <u>compact</u>, if each open cover of K has a finite subcover, i.e., $\exists\, \gamma_1, \dots, \gamma_n \in \Gamma : \{O_{\gamma_1}, \dots, O_{\gamma_n}\}$ is an open cover of K.

Example

$(0, 1]$ and $\{(1/k, 1 + 1/k) \mid k \in \mathbb{N}\}$ not compact

$O_k = \left(\frac{1}{k}, 1 + \frac{1}{k}\right)$

a finite subcollection would not cover points sufficiently close to 0: $k = \max(k_1, \dots, k_n)$, $\frac{1}{2k} \notin \bigcup_{i=1}^{n} O_{k_i}$.

Properties of Compact Sets in Metric Spaces

Theorem

If K is a compact subset of a metric space X, then K is closed and bounded.

Remark. The converse is true for $X = \mathbb{R}^n$ (the Heine-Borel thm)

Examples

$[0,1] \subset \mathbb{R}$ compact

closed ball $\overline{B_\delta(x)} \subset \mathbb{R}^n$ compact

$\overline{B_\delta(x)} = \{y \in X \mid d(x,y) \leq \delta\}$