Math 5615H: Honors: Introduction to Analysis
Compact Sets

Sasha Voronov

University of Minnesota

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Properties of Compact Sets in Metric Spaces

Theorem

If K is a compact subset of a metric space X, then K is closed and bounded.

Remark. The converse is true for $X = \mathbb{R}^n$ (the Heine-Borel thm)

Examples

$[0,1]$, $[a,b] \subset \mathbb{R}$

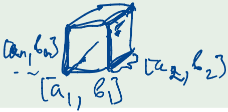
$\overline{B_\delta(x)} \subset \mathbb{R}^n$,

$d(a,b) \le 2\delta$

an n-cell in $\mathbb{R}^n$

$[a_1,b_1] \times [a_2,b_2] \times \dots \times [a_n,b_n]$

$\subset \mathbb{R}^n$

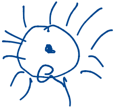


Proof of Theorem

Suppose K compact $\subset X$.

Bounded: Cover X (and thereby $K \subset X$) with open balls $B_n(x_0)$ of radius $n \in \mathbb{N}$, centered at $x_0 \in X$. $\forall x \in X$ $\quad x_0 \quad x \quad d(x, x_0) < n \Rightarrow B_n(x_0) \ni x$

Closed: Prove $K^c := X \cancel{\setminus} K$ is open. Given $a \in K^c$, consider $O_k = \{x \in \cancel{X} \mid \cancel{|x - a|}\, d(x,a) > 1/k\}$, $k \in \mathbb{N}$. Then $\bigcup_{k=1}^{\infty} O_k = X \setminus \{a\}$, so $K \subset \bigcup_{k=1}^{\infty} O_k$. b/c $K \subset X \setminus \{a\}$



Pick a finite subcover

$$\bigcup_{i=1}^{n} O_{k_i} = O_{k_p}$$

where p is such that $k_p = \max\{k_1, \dots, k_n\}$

$$K \subset O_{k_p} \Rightarrow K^c \supset O_{k_p}^c$$

$$B_{1/k_p}(a) \subset O_{k_p}^c \subset K^c, \quad \square$$

Closed Subsets of Compact Sets

Theorem

Let K be a closed subset of a compact metric space X. Then K is compact. In particular, a closed subset of a compact set in any metric space is compact. $X \subset X$

$K \subset X$
closed compact

Proof. Let $\{O_\alpha\}$ be an open cover of K. Then

$$\{K^c\} \cup \{O_\alpha\}$$

open, b/c K closed

is an open cover of X.

$\bigcup_\alpha O_\alpha \supset K$, $\left(\bigcup_\alpha O_\alpha\right) \cup K^c \supset X$.

Choose a finite subcover $\{O_{\alpha_i} \mid i = 1, \dots, n\}$ $\cup \{K^c\}$

Drop $\{K^c\}$ ~~of~~ with or without this anyway. $\bigcup_{i=1}^n O_{\alpha_i} \cup K^c \supset X \Rightarrow K \subset \bigcup_{i=1}^n O_{\alpha_i}$ $\square$

The Bolzano-Weierstrass Property

Definition

Let X be a metric space. A subset $A \subset X$ has the *Bolzano-Weierstrass property* if every infinite subset of A has a limit point (cluster point) that belongs to A.

Theorem

Let A be a subset of a metric space (X, d). Then A is compact iff A has the Bolzano-Weierstrass property.

Proof. Today only: A compact $\Rightarrow$ A has B-W property

$E \subset A$ infinite. Suppose ~~Every~~ no point of A is a limit point of E. $\Rightarrow \forall a \in A$ $\exists$ open ball B_a centered at a: $B_a \cap E = \begin{cases} \{a\} \text{ or} \\ \emptyset \end{cases}$. This gives an open cover $\{B_a \mid a \in A\}$ of A. A compact $\Rightarrow \exists$ finite subcover. It covers A and thereby E. $\Rightarrow E$ finite. $\square$

Next time $\Leftarrow$

Nested Intervals

Theorem (Nested interval property)

~~**Proof.**~~ $X = \mathbb{R}^n$, n-cells $I_k = [a_1^{(k)}, b_1^{(k)}] \times [a_2^{(k)}, b_2^{(k)}] \times \dots \times [a_n^{(k)}, b_n^{(k)}] \subset \mathbb{R}^n$

Suppose those I_k form a nested sequence of n-cells, i.e.,

$$I_1 \supset I_2 \supset I_3 \supset \dots \supset I_k \supset I_{k+1} \supset \dots$$

Then $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$. If $|b_j^{(k)} - a_j^{(k)}| < \frac{1}{k}$ $\forall j = 1, \dots, n$

Then $\bigcap_{k=1}^{\infty} I_k$ is a single point.

Proof: next time.

The Heine-Borel Theorem

Theorem

A subset K of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof.

Next time