Proposition 0.0.1. ***Problem 6 HW6***
Let (p_n) be a sequence of real numbers. We have that (p_n) is bounded above if and only if $\limsup_{n\to\infty} p_n < \infty$.

Remark: Absolutely direct. No contraposition. No contradiction.

Remark: The following statements are equivalent: (i) $\sup(A) \leq M$ and (ii) $x \leq M$ for every $x \in A$.

Remark: Nobody gave a proof even remotely like what I have called "using Rudin's definition." Rather, almost everyone proved a contrapositive. Note that to do so is fine, but it's also satisfying to see a direct proof, no?

Proof. (*using Rudin's definition*) On the one hand, suppose that the sequence of real numbers (p_n) is bounded above. Let E denote the set of sub-sequential limits of (p_n). [1] In order to show that $\sup(E) < \infty$, it suffices to show that there exists an $M \in \mathbb{R}$ such that $x \leq M$ for every $x \in E$. For concreteness, let M be an arbitrary upper bound of (p_n). Thus, $p_n \leq M$ for all n. Next, give me any $x \in E$; I will show that $x \leq M$. By definition of E, there exists a sub-sequence (p_{n_k}) converging to x. Because $M \geq p_n$ for every n, we have $M \geq p_{n_k}$ for every k. Hence, letting $k \to \infty$, [2] we have $M \geq x$, as required. Overall, by the preceding comments, this means $\sup(E) \leq M$ and, so, $\sup(E) < \infty$.

Conversely, suppose that $\limsup_{n\to\infty} p_n < \infty$. Then, because $\mathbb{R}$ is Archimedean, there exists a real number L such that $\limsup_{n\to\infty} p_n < L$. Consider any such L. Theorem 3.17 (b) of Rudin states that, as a result, we have $p_n \leq L$ for all $n \geq N$ for some N. Then, the sequence (p_n) is bounded above by $\max\{p_1, \ldots, p_N, L\}$. □

Proof. (*using Problem 5*) On the one hand, suppose that the sequence of real numbers (p_n) is bounded above. Thus, consider M such that $p_m \leq M$ for every m. Then, in particular, $p_m \leq M$ for every $m \geq n$, for any n. Equivalently, $\sup_{m\geq n}\{p_n\} \leq M$ for any n. Letting $n \to \infty$, this implies $\limsup_{m\geq n}\{p_n\} = \lim_{n\to\infty} \sup_{m\geq n}\{p_n\} \leq M$, as desired.

Conversely, suppose only that $\limsup_{m\geq n}\{p_n\} < \infty$. If (p_n) were bounded above, then we would unambiguously be justified in employing Problem 5. However... the fact that (p_n) is bounded above is exactly what we are now trying to prove. So, this should at least make us uncomfortable. [3] We should be more careful, if we can. Actually, for this part, we can use the exact same argument as in the preceding proof! Phew. □

[1] For what it's worth, we have a theorem which says that this set is non-empty.

[2] What does this mean? We are employing the following fact and trusting the reader to understand: If $\lim_{k\to\infty} a_k = a$ and $a_k \leq L$ for each k, then $a \leq L$. This is a general property of convergent sequences of real numbers.

[3] Note that the result in Problem 5 is, in fact, true *without* assuming that the sequence is bounded. However, there are subtleties which arise. For instance, when (p_n) is not bounded, it is possible that a set of the form $\{p_n : n \geq m\}$ will have infinite supremum, in which case our sequence $\sup_{n\geq m}\{p_n\}$ of which we want to take the limit has ... extended real numbers in its range. This is new. In fact, such sequences are frequently encountered in, e.g., measure theory. But we have not encountered them yet. OK, you say, I don't care about that. I should be fine because, *if* $\sup_{n\geq m}\{p_n\} \to L$ as $m \to \infty$, *then* $\sup_{n\geq m}\{p_n\} \in \mathbb{R}$ for every m. This is true, too. The problem is that this is an even stronger variation of the result which we are trying to prove. So, we don't really make any progress when going down this rabbit hole.