Math 5615H **Fall 2020** **Homework 10**

Posted: 11/21; A minor typo corrected: 11/23; due Wednesday, 12/02

The problem set is due at the beginning of the class on Wednesday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**).**
(Baby) Rudin: Sections 5.1-5.15.

Problem 1. Give the critique of (*i.e.*, find a flaw in) the following plausible "proof" of the chain rule:

$$\lim_{t\to x}\frac{g(f(t))-g(f(x))}{t-x}=\lim_{t\to x}\frac{g(f(t))-g(f(x))}{f(t)-f(x)}\frac{f(t)-f(x)}{t-x}$$
$$=\left(\lim_{t\to x}\frac{g(f(t))-g(f(x))}{f(t)-f(x)}\right)\left(\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\right)$$
$$=g'(f(x))f'(x).$$

Problem 2. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a function whose derivative exists at each point and is bounded. Show that f is uniformly continuous.

Problem 3. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2\sin(1/x)+x/2$ for $x\neq 0$ and $f(0)=0$. From the previous homework, we easily conclude that f is differentiable on $\mathbb{R}$. Show that the derivative of f at 0 is >0 but there is no $\delta>0$ such that f is increasing in the interval $(-\delta,\delta)$. Does it contradict the theorem about functions with positive derivative being increasing, see Class Notes of 11/20 (or Theorem 5.11 from Rudin)?

Problem 4.
(1) Suppose that $f:\mathbb{R}\to\mathbb{R}$ is continuous and
$$\lim_{|x|\to+\infty}f(x)=0.$$
Prove that f is uniformly continuous.
(2) Find a bounded function $f:\mathbb{R}\to\mathbb{R}$ such that f is differentiable at every point and uniformly continuous, but f' is not bounded. *Hint*: Make use of Part (1). Note that the derivative must oscillate between large positive and negative values as $|x|\to\infty$, because if, say, f' just grows monotonically and unboundedly with $x\to+\infty$, then it should force f to do the same. For instance, try to see why $\sin x/x$ does not work and cook something based on that.

Problem 5. Suppose that f is differentiable at each point of (a,b) and its derivative is never 0. Prove that f is strictly increasing or strictly decreasing on the interval. (Note that f' is not assumed to be continuous.)

Problem 6. Suppose f is a real-valued function on $(0, +\infty)$ with the properties:

(1) $f(xy) = f(x) + f(y)$ for all positive x and y;
(2) $f'(1)$ exists and equals 1.

Prove that $f(1) = 0$ and $f'(x)$ exists and equals $1/x$ for all $x > 0$. *Hint*: For the second statement, do $x + h = x(1 + h/x)$.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that $f^{(n)}(0) = 0$ for all $n \geq 0$. *Hint*: Use induction on n and L'Hôpital's rule after a change of variables $y = 1/h$.

Problem 8. Prove that $\lim_{x \to +\infty} x^n e^{-x} = 0$ for every $n \in \mathbb{N}$ in two ways: using l'Hôpital's rule and Taylor's theorem.