**Math 5615H** **Fall 2020** **Homework 11**
**Posted: 12/2; Problem 8 corrected 12/9; due Wednesday, 12/9**

The problem set is due at the beginning of the class on Wednesday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

**Reading**: **Class notes (available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**).** **(Baby) Rudin**: Sections 5.14-5.15, 9.22-9.23, and Exercises 5.22 and 5.25.

**Problem 1.** Suppose that $f : [0,1] \to \mathbb{R}$ is continuous and $f(0) = f(1) = 0$. Suppose that $f''$ exists and $f''(x) \geq 0$ at each point $x$ of $(0,1)$. Prove that $f(x) \leq 0$ for all $x \in (0,1)$.

**Problem 2.** Suppose that $f : (a,b) \to \mathbb{R}$ and $f''$ exists and $f''(x) \geq 0$ at each point $x$ of $(a,b)$. Prove that $f$ is convex, *i.e.*, $f(tx+(1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in (a,b)$ and each $t \in (0,1)$.

**Problem 3.** Suppose for some $n \geq 1$, the $n+1$st derivative of $f$ exists and is continuous on some open interval $(-a,a)$ with $a > 0$. Suppose also that there is a polynomial $p(x)$ of degree $\leq n$ such that

$$|f(x) - p(x)| \leq C|x|^{n+1}$$

for some $C > 0$ and all $x \in (-a,a)$. Prove that the polynomial $p(x)$ is the $n$th Taylor polynomial for $f$ centered at 0. That is, prove that

$$p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

**Problem 4.** Suppose that $f$ is a bounded real-valued function on $\mathbb{R}$ with bounded and continuous first and second derivatives.

(1) Use Taylor's theorem around any fixed $x$ to conclude that for all $h > 0$,

$$|f'(x)| \leq \frac{2}{h} \sup_{x\in\mathbb{R}} |f(x)| + \frac{h}{2} \cdot \sup_{x\in\mathbb{R}} |f''(x)|.$$

(2) Prove that

$$\sup_{x\in\mathbb{R}} |f'(x)|^2 \leq 4 \sup_{x\in\mathbb{R}} |f(x)| \cdot \sup_{x\in\mathbb{R}} |f''(x)|$$

by choosing the best $h$ in Part (1).

**Problem 5** (Tianyi Wei)**.** Show that every contraction mapping $T : X \to X$ on an arbitrary metric space $X$ is uniformly continuous.

**Problem 6.** Suppose $f$ is three times continuously differentiable. Let $h = x - a$ for $x$ near $a$. By Taylor's theorem, for sufficiently small $h$, we have

$$f(a+h) - f(a) = f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f^{(3)}(c)}{3!}h^3$$

for some $c$ between $a$ and $x$. Establish the following using Taylor's theorem.

(1) Show that if there is a local extremum of $f$ at $x = a$, then $f'(a) = 0$. (We know this from an earlier theorem, but derive it now from the expression above by analyzing the behavior of the left-hand and right-hand sides for small enough $h$.)
(2) Show (similarly, just working with the equation above) that if there is a local extremum of $f$ at $a$ and $f''(a) > 0$, then $f$ has a local minimum at $a$.
(3) Show similarly that the condition $f''(a) < 0$ is sufficient for a local extremum to be a local maximum.

**Problem 7.** Suppose we want to approximate a solution to $f(x) = x^3 - x - 1 = 0$ on $[1, \infty)$. We rewrite the equation in an equivalent form $x = (x+1)^{1/3} =: g(x)$. Show that $g(x)$ is a contraction mapping on $[1, \infty)$. Will the iterations $x_{n+1} = g(x_n)$ with any $x_1 \geq 1$ converge to the root of $f(x) = 0$?

**Problem 8.** Let $f : [1, \infty) \to [1, \infty)$ be defined by $f(x) = x + e^{1-x}$.

(1) Show that $|f(x) - f(y)| < |x - y|$ for all $x \neq y$ in $[1, \infty)$.
(2) Show that $f$ has no fixed point in $[1, \infty)$.