Math 5615H **Homework 2**
Posted: 09/19; due Friday, 09/25

The problem set is due at the beginning of the class on Friday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (some available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**). (Baby) Rudin**: Sections 1.7-1.11, 1.17-1.21, 1.24-1.33.
Problems:

1. Prove that there is no way to introduce a positive set in (or, equivalently, a total order on) the field $\mathbb{C}$ of complex numbers so that it becomes an ordered field. *Hint*: Shall we have $i > 0$ or < 0?

2. Show that in an ordered field, the elements $n \cdot 1$, $n \in \mathbb{N}$, that is, elements 1, $1+1$, $1+1+1$, $\dots$, are all positive and distinct. (This implies that every ordered field is infinite.)

3. Without assuming that $\sqrt{12}$ and $\sqrt{x}$ exist, show that if $x > 0$ is real and $x^2 < 12$, then there is a rational $y > x$ such that $y^2 < 12$. *Hint*: Show that if $h > 0$ is a sufficiently small, then $(x+h)^2 < 12$. Then use Theorem 1.20(b).

4. Show that any automorphism of the field $\mathbb{R}$ of real numbers is trivial, i.e., the identity. An automorphism is a bijection $\mathbb{R} \to \mathbb{R}$ which respects addition and multiplication, as well as the zero and unit elements, 0 and 1. (By the way, since every complete ordered field is isomorphic to $\mathbb{R}$, this implies that an isomorphism between two complete ordered fields is unique.) *Hint*: First, show that the rationals must be fixed by an automorphism. Then show that an automorphism must preserve the ordering of the real numbers. (Surprise: algebra enforces analysis!)

5. Prove that the field of complex numbers is not isomorphic to the field of real numbers.

6. Define e^{it} for real t as $\cos t + i \sin t$. Show that $|e^{it}| = 1$ and $e^{is+it} = e^{is}e^{it}$. Show that every nonzero complex number can be factored into re^{it} with $r, t \in \mathbb{R}$ and $r > 0$. This is called *polar decomposition*.

7. Show, using polar decomposition, that for any $n \in \mathbb{N}$, there are exactly n solutions to the equation $z^n = 1$.

8. Let S be a bounded above subset of an ordered field F. Prove the following characterization of supremum $\sup S$. An upper bound b of S is the least upper bound of S, if and only if for every positive element $\varepsilon \in F$, there is an element $x \in S$ such that $b - \varepsilon < x \le b$.