Math 5615H **Homework 3**

Posted: 09/26; numbering and Problem 9 corrected 10/1; due Friday, 10/2

The problem set is due at the beginning of the class on Friday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (some available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**).**
(Baby) Rudin: Sections 1.22, 1.34-1.38, 2.14-2.27, 2.31-2.35.

Problem 1. Use the notation $2^{|A|} := |P(A)|$, the cardinality of the set $P(A)$ of all subsets of A, for any set A. Oftentimes $\aleph_0 := |\mathbb{N}|$ denotes the cardinality of the naturals, and the sequence taking $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, ... defines a countable hierarchy of increasing cardinalities, as we can easily see from the previous homework. ($\aleph$ (aleph) is the first letter of the Hebrew alphabet.) The *cardinality of continuum* $\mathfrak{c}$ is, by definition, the cardinality of the reals: $\mathfrak{c} := |\mathbb{R}|$. Show that $\mathfrak{c} = 2^{\aleph_0}$. *Hint*: For any set A and a subset $B \subset A$, let $\chi_B : A \to \{0, 1\}$ be the *characteristic function* of B:

$$\chi_B(a) := \begin{cases} 1, & \text{if } a \in B, \\ 0, & \text{if } a \notin B. \end{cases}$$

Use characteristic functions to find a bijection between the set $P(A)$ of subsets of A and the set of functions $A \to \{0, 1\}$. Then ask yourself what a function $\mathbb{N} \to \{0, 1\}$ reminds you of?

Problem 2. If $|A_n| = \mathfrak{c}$ for all $n \geq 1$ under the notation of the previous problem, then the countable disjoint union $\coprod_{n\geq 1} A_n$ has the same cardinality $\mathfrak{c}$.

Problem 3. Show that in a discrete metric space, each subset is open and closed. (A metric space is called *discrete* if $d(x, y) = 1$ whenever $x \neq y$).

Problem 4. Find an infinite collection of distinct open sets in $\mathbb{R}$ whose intersection is a nonempty open set. (Thus infinite intersections of open sets may happen to be open.)

Problem 5. Show that $\mathbb{Q}$ as a subset of $\mathbb{R}$ is neither open, nor closed.

Problem 6. Show that the closure $\overline{S}$ of a subset S of a metric space is the intersection of all the closed sets which contain S.

Problem 7. Let A be a subset of a metric space X. Show that $\overline{A} = A \cup \partial A$, where $\overline{A}$ is the closure and ∂A is the boundary of A.

Problem 8. Let A be a subset of a metric space X and let x_0 be an isolated point of A. Show that x_0 is in the boundary of A if and only if x_0 is a cluster (limit) point of A^c.

Problem 9. A metric space X is called *separable* if there exists an at most countable subset of X which is dense in X. Show that if X is compact, then it is separable. *Hint*: For each fixed $n \in \mathbb{N}$, consider the covering of X by the open balls of radius $1/n$ centered at every point of X.