Math 5615H **Fall 2020** **Homework 5**

Posted: 10/16; Problem 1 reworded: 10/21; due Friday, 10/23

The problem set is due at the beginning of the class on Friday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (some available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**).**
(Baby) Rudin: Sections 3.3-3.14, 3.20-3.26, 3.28, 3.30-3.32, 3.47.

Problem 1. Guess and write the definition of an *eventually constant* sequence in a metric space. Let $\{p_n\}$ be a sequence in a *discrete* metric space X. This is a metric space with the distance between any two distinct points is equal to 1. Show that any Cauchy sequence in a discrete X is eventually constant.

Problem 2. Is any discrete metric space complete? Explain your answer.

Problem 3 (The squeeze theorem)**.** Let $\{a_k\}$, $\{b_k\}$, and $\{c_k\}$ be sequences of real numbers such that for each k, $a_k \le b_k \le c_k$. If $\lim_{k\to\infty} a_k = \lim_{k\to\infty} c_k = L$ for some real number L, then $\lim_{k\to\infty} b_k = L$.

Problem 4. If $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a metric space, show that for any $\epsilon > 0$ there exists a subsequence $\{p_{n_i}\}_{i\in\mathbb{N}}$ so that $d(p_{n_i}, p_{n_{i+1}}) < \epsilon/2^{i+1}$.

Problem 5. Show that any sequence $\{s_n\}$ in $\mathbb{R}$ has a monotone subsequence. *Hint*: Start with supposing that it does not have a monotone increasing subsequence.

Problem 6. Let z be a complex number, $|z| < 1$. Prove that $\lim_{n\to\infty} z^n = 0$, using the ϵ-N definition of a limit. *Hint*: Prove and use the inequality $(1+h)^n \ge 1 + nh$ for any positive h and natural n. See also the (very sketchy) proof of Theorem 3.20 (e), in which x is real.

Problem 7. Suppose that $\{s_n\}$ is the sequence of partial sums of a series $\sum a_n$. Suppose that $\lim_{n\to\infty} a_n = 0$ and that $\lim_{n\to\infty} s_{2n}$ exists. Prove that the series converges.

Problem 8. Let $x \in \mathbb{R}$, $x > 0$. Define $e^x := \lim_{n\to\infty}(1 + \frac{x}{n})^n$ (assume the limit exists). Prove that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.