Math 5615H **Fall 2020** **Homework 8**

Posted: 11/7; Problems added 11/7; due Friday, 11/13

The problem set is due at the beginning of the class on Friday. Please scan handwritten pages and upload the resulting image file or the pdf file produced by LaTeX of whatever document preparation software you use to Canvas.

Reading: **Class notes (available on the Course Outlines page** `http://www-users.math.umn.edu/~voronov/5615-20/outline.html`**).**
(Baby) Rudin: Sections 4.1-4.16, 4.18-4.19, 4.22-4.24.

Problem 1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ (reduced fraction)}, x \neq 0, \\ 0 & \text{if } x = 0 \text{ or } x \notin \mathbb{Q}. \end{cases}$$

Show that f is continuous at 0 and any irrational point, but not continuous at any nonzero rational point.

Problem 2. Describe all continuous functions $f : \mathbb{R} \to X$, where X is a discrete metric space.

Problem 3. Let S be a metric space and $q \in S$. Show that the distance function $d(p, q)$ is a continuous function of p.

Problem 4. Let E be a nonempty subset of a metric space S. Define the distance from a point $p \in S$ to the set E to be

$$d_E(p) = \inf\{d(p, q) \mid q \in E\}.$$

Prove that $d_E(p) = 0$ iff $p \in \overline{E}$, the closure of E. Prove that d_E is a continuous function on S.

Problem 5. Suppose that E is a subset of a metric space S that is not closed. Show that there is a continuous real-valued function on E that is not bounded.

Problem 6. Prove the squeeze theorem worded in class on 11/6/2020.

Problem 7. Let $p : \mathbb{R} \to \mathbb{R}$ be a nonconstant polynomial function and $K \subset \mathbb{R}$ be a compact set. Show that $p^{-1}(K)$ is compact. You do not need to prove that polynomials are continuous. (They are, as linear combinations of powers of the identity function $p(x) = x$.)