Math 5615H

Homework 11

Posted: 11/22; Updated: 11/26; due: Monday, 12/1/2014

The problem set is due at the beginning of the class on Monday after the Thanksgiving Break.

Reading: Chapter 5: pages 109-113.

For this homework, you may assume that \((\sqrt{x})' = 1/(2\sqrt{x})\), \(\sin' x = \cos x\) and similar computations known to you from calculus.

Problem 1. (1) \(\lim_{x \to 1} \frac{x \log x}{e^x - e}\);
(2) \(\lim_{x \to 0} x^{-a} \log x, \quad a > 0\);
(3) \(\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}\), provided \(f''(x)\) exists and is continuous at and near \(x\).

Problem 2. Let \(f : \mathbb{R} \to \mathbb{R}\) be defined by
\[
f(x) = \begin{cases} 
e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}
\]
Show that \(f^{(n)}(0) = 0\) for all \(n \geq 0\). Hint: Use induction on \(n\) and L’Hôpital’s rule after a change of variables \(y = 1/h\).

Problem 3. Find the Taylor expansions of the following functions \(f(x)\) about the indicated points to the indicated term:
(1) \(\sin x\) about \(x_0 = \pi\) to the general, \(n\)th term;
(2) \(\frac{1}{x-1}\) about \(x_0 = -1\) to the \(n\)th term;
(3) \(e^{-1/x^2}\), \(f(0) = 0\), about \(x_0 = 0\) to the \(n\)th term;
(4) \(\sqrt{x^2 + 1}\) about \(x_0 = 2\) to the sixth term.

Problem 4. (1) Suppose that \(f''\) exists and is continuous on \((a, b)\), \(f'(x_0) = 0\) and \(f''(x_0) > 0\) for some \(x_0 \in (a, b)\). Show that \(f\) has a local minimum at \(x_0\).
(2) Suppose that \(f^{(n)}\) exists and is continuous on \((a, b)\), \(f^{(k)}(x_0) = 0\) for \(1 \leq k \leq n - 1\) and \(f^{(n)}(x_0) \neq 0\) for some \(x_0 \in (a, b)\). Under what conditions on \(n\) can you say that \(f\) must have a local extremum at \(x_0\)?

Problem 5. Prove that \(\lim_{x \to +\infty} x^n e^{-x} = 0\) for every \(n \in \mathbb{N}\) in two ways: using l’Hôpital’s rule and Taylor’s theorem.

Problem 6. Suppose that \(f : [0, 1] \to \mathbb{R}\) is continuous and \(f(0) = f(1) = 0\). Suppose that \(f''\) exists and \(f''(x) \geq 0\) at each point \(x\) of \((0, 1)\). Prove that \(f(x) \leq 0\) for all \(x \in (0, 1)\).

Problem 7. Suppose that \(f : (a, b) \to \mathbb{R}\) and \(f''\) exists and \(f''(x) \geq 0\) at each point \(x\) of \((a, b)\). Prove that \(f\) is convex, i.e., \(f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)\) for all \(x, y \in (a, b)\) and each \(t \in (0, 1)\).