Math 5615H **Homework 11**

Posted: 11/22; Updated: 11/26; due: Monday, 12/1/2014

The problem set is due at the beginning of the class on Monday after the Thanksgiving Break.

Reading: Chapter 5: pages 109-113.

For this homework, you may assume that $(\sqrt{x})' = 1/(2\sqrt{x})$, $\sin' x = \cos x$ and similar computations known to you from calculus.

Problem 1. (1) $\lim\limits_{x\to 1} \dfrac{x \log x}{e^x - e}$;

(2) $\lim\limits_{x\to 0} x^{-a} \log x, \qquad a > 0$;

(3) $\lim\limits_{h\to 0} \dfrac{f(x+h)+f(x-h)-2f(x)}{h^2}$, provided $f''(x)$ exists and is continuous at and near x.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that $f^{(n)}(0) = 0$ for all $n \geq 0$. *Hint*: Use induction on n and L'Hôpital's rule after a change of variables $y = 1/h$.

Problem 3. Find the Taylor expansions of the following functions $f(x)$ about the indicated points to the indicated term:

(1) $\sin x$ about $x_0 = \pi$ to the general, nth term;

(2) $\frac{1}{x-1}$ about $x_0 = -1$ to the nth term;

(3) e^{-1/x^2}, $f(0) = 0$, about $x_0 = 0$ to the nth term;

(4) $\sqrt{x^2+1}$ about $x_0 = 2$ to the sixth term.

Problem 4. (1) Suppose that f'' exists and is continuous on (a, b), $f'(x_0) = 0$ and $f''(x_0) > 0$ for some $x_0 \in (a, b)$. Show that f has a local minimum at x_0.

(2) Suppose that $f^{(n)}$ exists and is continuous on (a, b), $f^{(k)}(x_0) = 0$ for $1 \leq k \leq n-1$ and $f^{(n)}(x_0) \neq 0$ for some $x_0 \in (a, b)$. Under what conditions on n can you say that f must have a local extremum at x_0?

Problem 5. Prove that $\lim_{x\to+\infty} x^n e^{-x} = 0$ for every $n \in \mathbb{N}$ in two ways: using l'Hôpital's rule and Taylor's theorem.

Problem 6. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and $f(0) = f(1) = 0$. Suppose that f'' exists and $f''(x) \geq 0$ at each point x of $(0, 1)$. Prove that $f(x) \leq 0$ for all $x \in (0, 1)$.

Problem 7. Suppose that $f : (a, b) \to \mathbb{R}$ and f'' exists and $f''(x) \geq 0$ at each point x of (a, b). Prove that f is convex, *i.e.*, $f(tx+(1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in (a, b)$ and each $t \in (0, 1)$.