Math 5615H

Homework 4
Posted: 11:30 a.m., 09/27; modified: 10/01; due: Friday, 10/3/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 2: pages 35-43.

Problems:

1. Let $A$ be a subset of a metric space $X$ and let $x_0$ be an isolated point of $A$. Show that $x_0$ is in the boundary of $A$ if and only if $x_0$ is a limit point of $A^c$.

2. Let $X$ be a metric space. If $A \subset X$ has the property that every infinite subset of $A$ has a limit point in $A$, show that for any open covering there exists a countable or finite subcovering. Hint: First, construct a countable base for $A$, that is, a countable collection $\{U_i\}_{i \in \mathbb{N}}$ of open sets in $X$, such that if $V$ is an open set in $X$ and $x \in A \cap V$, then there exists some $i \in \mathbb{N}$ such that $x \in U_i \subset V$. To construct such a base, show for each $n \in \mathbb{N}$, there is a finite number of balls of radius $1/n$ covering $A$. Use the constructed base to select a countable (or, if you are lucky, finite) subcovering of a given open covering $\{V_\alpha\}_{\alpha \in I}$ of $A$.

3. Let $A$ be a subset of a metric space $X$. Show that $\overline{A} = A \cup \partial A$, where $\overline{A}$ is the closure and $\partial A$ is the boundary of $A$.

4. Show that in a discrete metric space, each subset is open and closed.

5. A metric space $X$ is called separable if there exists a countable subset of $X$ which is dense in $X$. Show that if $X$ is compact, then it is separable. Hint: For each fixed $n \in \mathbb{N}$, consider the covering of $X$ by the open balls of radius $1/n$ centered at every point of $X$.

6. Show that the (open) unit ball $B_1(0)$ about the origin $0$ in $\mathbb{R}^k$ with the “taxicab” metric is convex.

7. Show that $\mathbb{R}^k$ is separable in the sense of Problem 5 above.

8. Show that any convex subset of $\mathbb{R}^k$ is connected.

9. Show that the Cantor set consists of all the numbers in the closed interval $[0, 1]$ whose ternary expansion has only 0’s and 2’s and may end in infinitely many 2’s.