Math 5615H **Homework 7**

Posted: 10/25; Updated 10/30; due: Friday, 10/31/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 3: pages 65-75.

Problem 1. Suppose that b is positive. Compute

$$\lim_{n\to\infty} n\left(\frac{(n+1)^b}{n^b}-1\right).$$

You may use L'Hôpital's rule or see the class news page on the web to see how you can do the problem without L'Hôpital's help. You may also use the squeeze theorem: if $\{a_n\}$ and $\{c_n\}$ converge to the same limit L and $a_n \le b_n \le c_n$ for all large enough n, then $\{b_n\}$ converges to L. This theorem easily follows from the Cauchy criterion and the ability to pass to inequalities in limits, which we looked at.

Problem 2. For which $p > 0$ does the series

$$\sum_{n=3}^{\infty} \frac{1}{n\log n(\log\log n)^p}$$

converge?

Problem 3. Determine whether the series converges or diverges:

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n-\sqrt{n}}{n^2+5n}$;

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$;

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$;

(4) $\displaystyle\sum_{n=2}^{\infty} (\log n)^{-\log n}$;

(5) $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{\log n}{\log(n+1)}\right)$. *Hint*: Do some algebra to compare the nth term with $1/(n+1)\log(n+1)$, using $(1+1/n) > e^{1/(n+1)}$, which follows from the series expansion of e^x. See also the comment to Problem 1 on the news web page.

Problem 4. Determine the radius of convergence:

(1) $\displaystyle\sum_{n=0}^{\infty} n^a z^n, \qquad a \in \mathbb{R}$;

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n z^n}{n^3}$;

(3) $\displaystyle\sum_{n=0}^{\infty} a^{n^2} z^n, \qquad a > 0;$

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n;$

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n.$

Problem 5. Is there a real constant a such that the following series converges?

$$1 + \frac{1}{\sqrt{3}} - \frac{a}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{a}{\sqrt{4}} + \dots$$

Hint: Group the terms in triples

$$\frac{1}{\sqrt{4n-3}} + \frac{1}{\sqrt{4n-1}} - \frac{a}{\sqrt{2n}}$$

and do the algebra to compare the size of each triple with $c/\sqrt{n}$ for n large enough and some c. This will work for most values of a. For other a's, do the same grouping and algebra, but the result will be different.

Problem 6. Suppose that $a_n > 0$ for all n. Prove that $\sum_{n=1}^{\infty} a_n$ converges if

$$\liminf_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1$$

and diverges if $n(a_n/a_{n+1} - 1) \leq 1$ for all $n \geq N$ for some N. *Hint*: For the divergence statement, compare a_n to $1/n$. For the convergence statement, note that for some $c > 1$, we have $n\left(\frac{a_n}{a_{n+1}} - 1\right) > c$ for large enough n. Use your answer in Problem 1 to see that for any choice of b: $1 < b < c$, we have $n\left(\frac{(n+1)^b}{n^b} - 1\right) < c$ for large enough n. Deduce a comparison of a_n with $1/n^b$ (up to a constant, for large enough n).

Problem 7. What can you say about the radius of convergence of the product of two power series? Formulate a statement and prove it. *Hint*: We had a convergence statement at the end of the class today, based on Mertens' theorem (3.50 in the text). This should give a lower bound on the radius of convergence of the product. Try to find an example in which you do not have a larger radius of convergence for the product. Then formulate your best statement and prove it.