Math 5615H **Homework 9**

Posted: 11/08; Updated 11/12; due: Friday, 11/14/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 4: pages 89-98.

Problem 1. Determine whether the following functions are uniformly continuous on $(0, +\infty)$. You do not need to explain why here, which is unusual, except for the first two functions. Just answer *yes* or *no* for the rest.

(1) $f(x) = \sin x$. Explain your answer;
(2) $f(x) = 1/x$. Explain your answer;
(3) $f(x) = \sqrt{x}$;
(4) $f(x) = \log x$;
(5) $f(x) = x \log x$.

Problem 2. Let S be a metric space and $q \in S$. Show that the distance function $d(p, q)$ is a *uniformly* continuous function of p.

Problem 3. Find a closed, bounded subset A of $\mathbb{Q}$ and a continuous function $f : A \to \mathbb{R}$ such that f is not bounded. Explain why it does not contradict the fact that the image of a compact set under a continuous function is compact. *Hint*: A could be the intersection of a real interval with $\mathbb{Q}$.

Problem 4. Suppose that X is a discrete metric space. Show that every function from X to another metric space is uniformly continuous.

Problem 5. Find and unbounded subset $A \subset \mathbb{R}$ (with the usual metric) such that every function from A to a metric space is uniformly continuous.

Problem 6. Suppose that $f : A \to \mathbb{R}$ is a uniformly continuous real-valued function on a subset A of a metric space X.

(1) Suppose that $\{p_n\}$ is a Cauchy sequence in A. Show that $\{f(p_n)\}$ is a Cauchy sequence in $\mathbb{R}$.
(2) Show that there is a continuous real-valued function $g : \overline{A} \to \mathbb{R}$ defined on the closure $\overline{A}$ of A such that, for each $p \in A$, $g(p) = f(p)$. Show that g is unique.

Problem 7. Use the Intermediate Value Theorem (4.23) to show the following.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := x^n$ for some $n \in \mathbb{N}$. If b is a positive real number, show that there exists a unique positive real number a such that $a^n = b$.
(2) Show that any polynomial of odd degree has a real root.

Problem 8. For the function in Problem 4 on Homework 8, classify its discontinuities. That is, determine if the discontinuities at rational points are of first or second kind. Explain your answer by stating whether the left and right limits at rational points exist and, if they do, what they are, as in Examples 4.27 in the text.

Problem 9. A function $f : X \to X'$ is called *proper*, if for any compact subset $K \subset X'$, the preimage $f^{-1}(K) \subset X$ is compact. Show that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is proper if and only if $\lim |f(x)| = +\infty$ as $x \to +\infty$ and $x \to -\infty$.