MATH 5615H: HONORS ANALYSIS
SAMPLE FINAL EXAM (PART I)
NOW, WITH SELECTED SOLUTIONS

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You may not use a calculator, notes, books, etc. Only the exam paper, scratch paper, and a pencil or pen may be kept on your desk during the test. You must show all work.

Good luck!

Problem 1. Let x_1 be a real number, $x_1 > 1$, and let $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that the sequence $\{x_n\}$ is monotone and bounded and find its limit.

Problem 2. Is there a metric space which is countable and compact?

Problem 3. Assume that $f(x)$ is defined a real-valued for $x > 0$. Consider two statements:

(1) For every $m \in \mathbb{N}$, $x > 1/m$ implies $f(x) < 1/m$.
(2) $x > 0$ implies $f(x) \leq 0$.

Prove that (1) implies (2).

Problem 4. Prove or disprove the following statements with a precise ε-δ argument for each.

(1) The function $f(x) = x$ is uniformly continuous for all real x.
(2) The function $g(x) = \sin x$ is uniformly continuous for all real x.

Solution. Answer: Uniformly continuous. Given an $\varepsilon > 0$, take $\delta = \min \varepsilon/2, \pi/2, 1 > 0$. Then for any h such that $|h| < \delta$, we have

- $|\sin h| \leq |h|$: $|\sin h|$ is the shortest distance from the point $|h|$ radians, which will be in the first quadrant, because $|h| < \pi/2$, on the unit circle to the x axis, whereas $|h|$ is the length of the path along the circle;
- $0 < 1 - \cos h = 2\sin^2(h/2) \leq h^2/2 < |h|/2$, because $|h| < 1$.

Thus, $|\sin(x + h) - sinx| = |\sin x(\cos h - 1) - \sin h \cos x| \leq |\sin x| \cdot |\cos h - 1| + |\sin h| \cdot |\sin x| \leq 1 - \cos h + |\sin h| < |h|/2 + |h| < \varepsilon/4 + \varepsilon/2 < \varepsilon$. □

(3) The function $(f \cdot g)(x) = x \sin x$ is uniformly continuous for all real x.

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Solution. Answer: Not uniformly continuous. Take $\varepsilon = 1$. Then for any given $\delta > 0$, take $\delta_1 = \min(\pi/2, \delta/2) > 0$ and $n \in \mathbb{N}$ such that $2\pi n \sin \delta_1 > 1$. Such n exists because we can apply the Archimedean principle to $2\pi \sin \delta_1$ which is > 0, given that we made sure that $0 < \delta_1 < \pi/2$. Then take $x = 2\pi n$ and $y = 2\pi n + \delta_1$. Then $|x - y| = \delta_1 < \delta$, but $|x \sin x - ysiny| = |y \sin y| = (2\pi n + \delta_1) \sin \delta_1 > 2\pi n \sin \delta_1 > 1$. □

Problem 5. Prove that if a power series $\sum_{n=0}^{\infty} c_n z^n$ converges for some $z = z_0 \neq 0 \in \mathbb{C}$, then $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely for all $z \in \mathbb{C}$ with $|z| < |z_0|$. What does this say about the radius of convergence of the series? Use this to show that the radius of convergence of the exponential series $\sum_{n=0}^{\infty} z^n/n!$ is $+\infty$.

Solution. Apply the Root **Divergence** Test (Theorem 3.33 (b), not (a) or (c)!) to the convergent series $\sum_{n=0}^{\infty} c_n z_0^n$:

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|}|z_0| \leq 1, \tag{0.1}$$

because otherwise the series would diverge. (We **had** to use lim sup, rather than lim, because lim does not always exist, whereas lim sup does, at least in the extended real system. This problem with lim would render the conclusion $\lim_{n\to\infty} \sqrt[n]{|c_n|}|z_0| \leq 1$ to be simply incorrect: how can you **dare to compare** something that does not always exist with number 1?) Well, anyway, if $|z| < |z_0|$, then $\limsup_{n\to\infty} \sqrt[n]{|c_n|}|z| < \limsup_{n\to\infty} \sqrt[n]{|c_n|}|z_0|$, whence $\limsup_{n\to\infty} \sqrt[n]{|c_n|}|z| < 1$ and thereby, applying the Root **Convergence** Test (Theorem 3.33 (a), finally!) to the series $\sum_{n=0}^{\infty}|c_n z^n|$, we see that the series $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely.

Remark. We could have directly used Theorem 3.39 about **divergence** of power series to conclude from the convergence of the power series at $z = z_0$ that $|z_0| \leq R$, where R is the radius of convergence. The same theorem would then imply that the power series converges absolutely for $|z| < |z_0|$, because in this case $|z| < R$. I have put together the above, longer argument, because it uses more elementary facts and is more instructive.

The radius of convergence is, by definition, $R = 1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$. Because of Inequality (0.1), we have $R \geq |z_0|$.

For the exponential series, apply the Ratio Test (Theorem 3.34 (a)) to see that the exponential series converges for any $z = z_0 \neq 0 \in \mathbb{C}$. The test applies, because

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|n! z_0^{n+1}|}{|(n+1)! z_0^n|} = \lim_{n\to\infty} \frac{|z_0|}{n+1} = 0 < 1,$$

meaning "exists and equals 0," as always. (This inequality yields $\limsup_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty}|a_{n+1}/a_n| < 1$.) Thus, by the above, the radius of convergence R of the exponential series is at least $|z_0|$ for any $z_0 \neq 0 \in \mathbb{C}$. The only extended real number which satisfies this is $R = +\infty$. □