MATH 5615H: ANALYSIS
A SOLUTION TO PROBLEM 6(2) ON HW 9

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Here is a solution of Problem 6(2) on Homework 9.

Solution: We define $g(q) := \lim_{p\to q} f(p)$ for each $q \in \overline{A}$. Since $f(p)$ is continuous on A, $g(p) = f(p)$ for each $p \in A$. The limit exists, because for any sequence $\{p_n\} \subset A$ converging to $q \in \overline{A}$, the sequence $\{f(p_n)\}$ will converge in $\mathbb{R}$, as $\{p_n\}$ is a Cauchy sequence, and so is $\{f(p_n)\}$ by Part (1).

We need to show that g is continuous on $\overline{A}$. Let us do it using the definition. For each point $q \in \overline{A}$, given $\varepsilon > 0$, we need to find a $\delta > 0$ such that $|g(p) - g(q)| < \varepsilon$ whenever $|p - q| < \delta$ and $p \in \overline{A}$. Start with a $\delta_1 > 0$ such that $|f(p') - g(q)| < \varepsilon/2$ whenever $0 < |p' - q| < \delta_1$ and $p' \in A$. Such δ_1 exists, because $g(q) = \lim_{p'\to q} f(p')$. Note also that if $|p' - q| = 0$, which may happen only when $q \in A$, we have $|f(p') - g(q)| = |f(q) - g(q)| = 0 < \varepsilon/2$.

Define $\delta := \delta_1/2$. Now for any $p \in \overline{A}$ such that $|p - q| < \delta$, we can find $\delta_2 > 0$ such that $|f(p') - g(p)| < \varepsilon/2$ whenever $0 < |p' - p| < \delta_2$ and $p' \in A$. Such δ_2 exists, because $g(p) = \lim_{p'\to p} f(p')$. Note also that if $|p' - p| = 0$, which may happen only when $p \in A$, we have $|f(p') - g(p)| = |f(p) - g(p)| = 0 < \varepsilon/2$.

Take any point $p' \in A$ such that $|p' - p| < \min(\delta, \delta_2)$. Such p' exists, because if $p \in A$, we can take $p' = p$. Otherwise, p is a limit point of A and there are points of A arbitrarily close to p. Then $|p' - q| \le |p' - p| + |p - q| < \delta + \delta = \delta_1$, and we have

$$|g(p) - g(q)| \le |g(p) - f(p')| + |f(p') - g(q)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $g(q)$ is continuous on $\overline{A}$.

Now, let us show the uniqueness of g. Indeed, if we have another continuous function g_1 on $\overline{A}$ extending f from A, then we have $g_1(p) = f(p) = g(p)$ for all $p \in A$. If $q \in A'$, then take a sequence $\{p_n\} \subset A$ such that $p_n \to q$. Then, since g and g_1 are continuous on $\overline{A}$, we have $\lim_{n\to\infty} g_1(p_n) = g_1(q)$ and $\lim_{n\to\infty} g(p_n) = g(q)$. However, since $p_n \in A$, we have $g_1(p_n) = f(p_n) = g(p_n)$ and thereby $g_1(q) = g(q)$.

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