Math 5616H Homework 1
Posted: 1/23; Updated: 1/24; Due: Friday, 1/30/2015

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 6: Everything through 6.6, then 6.8, 6.12(a)-(d), 6.13(b), and 6.20-22.

Problem 1. Prove Theorem 6.12(b) for Riemann-integrable functions \( f_1 \) and \( f_2 \) on an interval \([a, b]\). Assume \( \alpha(x) = x \).

Problem 2. Suppose that \( f \) is a continuous, real-valued function on an interval \([a, b]\). Prove that there exists \( x \in [a, b] \) such that \( \int_a^b f(x) dx = f(x)(b - a) \).

Problem 3. Let \( f : [0, 1] \to \mathbb{R} \) be defined by
\[
f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}
\]
Show that \( f \) is integrable on \([0, 1]\). Hint: Try to use uniform continuity of \( f(x) \) on an interval \([\varepsilon/4, 1]\) to find a partition \( P \) with \( U(P, f) - L(P, f) < \varepsilon \).

Problem 4. Suppose that \( f \) is continuous and nonnegative on the interval \([a, b]\) and \( \int_a^b f(x) dx = 0 \). Prove that \( f \equiv 0 \) on \([a, b]\).

Problem 5. Let \( f(1/n) = 1 \) for \( n \in \mathbb{N} \) and \( f(x) = 0 \) otherwise. Prove that \( f \) is integrable on \([0, 1]\).

Problem 6. Let \( f : [0, 1] \to \mathbb{R} \) be defined by
\[
f(x) = \begin{cases} 1/q, & \text{if } x = p/q > 0 \text{ (reduced rational fraction)}, \\ 0, & \text{if } x = 0 \text{ or irrational}. \end{cases}
\]
Show that \( f \) is integrable on \([0, 1]\). FYI: This is an example of a function that is integrable on an interval but not differentiable at any point thereof.

Problem 7. Suppose that \( f \) is continuous, nonnegative, and monotonically increasing on \([0, \infty)\). Prove that
\[
\int_0^x f(t) dt \leq xf(x)
\]
for all \( x \geq 0 \).

Problem 8. Prove that
\[
\lim_{N \to \infty} \sum_{n=-N}^N \left( \frac{1}{N + in} + \frac{1}{N - in} \right) = 2 \int_{-1}^1 \frac{dt}{1 + t^2}.
\]
Hint: Represent the left-hand side as a Riemann sum, which we will study on Wednesday.