Math 5616H **Homework 11**

Posted: 4/18; Updated: 4/19; Due: Friday, 4/24/2015

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 11: Review Sections 13-18, read Sections 19-25.

Conventions: For this homework, $(X, \mathfrak{M}, \mu)$ will be a measure space, unless stated otherwise. By definition, a *measurable set* is one from $\mathfrak{M}$ and a *measurable function* on X is such $f : X \to [-\infty, \infty]$ that $f^{-1}(a, \infty]$ is a measurable set.

Problem 1. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions on a measure space $(X, \mathfrak{M}, \mu)$. Prove that the set $\{x \in X \mid \lim_n f_n(x)$ exists in $[-\infty, \infty]\}$ is a measurable set, *i.e.*, belongs to the σ-algebra $\mathfrak{M}$.

Problem 2. Prove that the characteristic function $\chi_{[0,1]}$ is not equal almost everywhere to a continuous function on $\mathbb{R}$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that if f is differentiable on $\mathbb{R}$, then f' is Borel measurable, meaning the preimage of $(a, \infty]$ is a Borel set. *Hint*: Observe that f' is a pointwise limit of functions

$$f_n(x) = \frac{f(x + 1/n) - f(x)}{1/n}.$$

Problem 4. Prove that if $f : X \to [0, \infty]$ is a nonnegative measurable function and $A \subset B$ are measurable sets, then $\int_A f d\mu \le \int_B f d\mu$.

Problem 5. Prove that if $\int_E f d\mu = 0$ for some $f \ge 0$ and $E \in \mathfrak{M}$, then $f = 0$ almost everywhere. *Hint*: Let $A_n := \{x \in E \mid f(x) > 1/n\}$. Show that $\mu(A_n) = 0$, given the assumption. Deduce that $\mu(\bigcup_n A_n) = 0$.

Problem 6. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) := 0$ if x is rational, and $f(x) := d^2$ if x is irrational, where d is the first nonzero digit in the decimal expansion of x. Show that $\int_{[0,1]} f dm = 95/3$. *Hint*: Is not f simple?

Problem 7. Show that if $f : X \to [0, \infty]$ is measurable and bounded and $\int_X f d\mu < \infty$, then $\int_X f^2 d\mu < \infty$.