Math 5616H **Homework 2**

Posted: 1/31; Updated 2/4; Due: Friday, 2/6/2015

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 7 through 7.15, and 7.26-33 (Skip proofs in 7.31-33 for the time being).

Problem 1. Show that the set $\mathcal{B}(X, F)$ of bounded functions from a set X to $F = \mathbb{R}$ or $\mathbb{C}$ with the metric $d(f, g) := \sup_{x \in X} |f(x) - g(x)|$ is a metric space.

Problem 2. Show that a Cauchy sequence $\{f_n\}$ in $\mathcal{B}(X, F)$ converges pointwise to some function f without using Theorem 7.8.

Problem 3. Consider a sequence of functions $f_n(x) = x^n : [0, 1] \to [0, 1]$. Find the pointwise limit f of the sequence $\{f_n\}$ and show that f is not continuous.

Problem 4. Define a sequence of functions $f_n : (0, 1) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{q^n}, & \text{if } x = \frac{p}{q} \neq 0 \text{ (reduced)}, \\ 0, & \text{otherwise}, \end{cases}$$

for $n \in \mathbb{N}$. Find the pointwise limit f of the sequence and show that the sequence converges to f uniformly.

Problem 5. Define a sequence of continuous, monotonically increasing functions $c_n : [0, 1] \to [0, 1]$ inductively as follows. Let $c_0(x) := x$ and, for $n \geq 0$, set

$$c_{n+1}(x) := \begin{cases} \frac{1}{2} c_n(3x), & 0 \leq x \leq 1/3, \\ 1/2, & 1/3 \leq x \leq 2/3, \\ \frac{1}{2} c_n(3x - 2) + \frac{1}{2}, & 2/3 \leq x \leq 1. \end{cases}$$

Show that the sequence $\{c_n\}$ is Cauchy. *Remark*: The limit function $c(x)$ will exist and be continuous by Theorem 7.15. The limit function $c(x)$ is called the *Cantor function* and also known as the *Cantor staircase function* and *Devil's staircase* – beware! *Hint*: You may use the following statement, which follows from the inductive definition:

$$\sup_{x \in [0,1]} |c_{n+1}(x) - c_n(x)| = \frac{1}{2} \sup_{x \in [0,1]} |c_n(x) - c_{n-1}(x)| \qquad \text{for all } n \geq 1.$$

You do not have to prove this statement.

Problem 6. Show that polynomial functions in the space $\mathcal{C}([0, 1], \mathbb{R})$ of continuous functions from $[0, 1]$ to $\mathbb{R}$ separate points. Is the same true for polynomial functions with integer coefficients?

Problem 7. A *trigonometric polynomial* is a function from the unit circle $S^1 := \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ in the complex plane $\mathbb{C}$ to $\mathbb{C}$ of the form $f(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}$, where the coefficients a_k are in $\mathbb{C}$. Show that the set of trigonometric polynomials is uniformly dense in the space $\mathcal{C}(S^1, \mathbb{C})$ of continuous functions. *Hint*: *Uniformly dense* means that the uniform closure of the set of trigonometric polynomials is the whole space $\mathcal{C}(S^1, \mathbb{C})$. There were other equivalent wordings of this, when we discussed the Weierstrass theorem in class on Wednesday.