Problem 1. Using differentiation, determine the coefficients \( a_n, n \geq 0, \) of the power series whose sum is \( (1 - z)^{-2} \) for \( |z| < 1. \)

Problem 2. Prove that, for any integer \( k \geq 0, \)
\[
\sum_{n=0}^{\infty} \binom{n+k}{k} z^n = \frac{1}{(1-z)^{k+1}}, \quad |z| < 1.
\]

Problem 3. Suppose that \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) has a radius of convergence \( R > 0, \) and suppose that \( |z_0| = r < R. \) Define
\[
g(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R - r.
\]
Prove that \( g(z) \) is given by a convergent power series
\[
g(z) = \sum_{n=0}^{\infty} b_n z^n
\]
whose radius of convergence is at least \( R - r. \) *Hint:* Use the Binomial theorem to compute the coefficients \( b_n \) and then use the previous problem to estimate the lim sup formula for the radius of convergence of \( g \) using the radius of convergence \( R \) of \( f. \)

Problem 4. Determine the coefficients of the power series that defines a function with the following properties: \( f''(z) = -f(z), \) \( f(0) = 1, \) \( f'(0) = 0. \)

Problem 5. Show that \( e^{z_1} = e^{z_2}, \) where \( e^z := E(z), \) if and only if \( z_1 - z_2 = 2\pi ni \) for some \( n \in \mathbb{Z}. \)

Problem 6. Extend \( C(x) \) and \( S(x) \) from (46) in the textbook to complex \( z: \)
\[
\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).
\]
Is it true that for each \( w \in \mathbb{C} \) there is \( z \in \mathbb{C} \) such that \( \cos z = w? \) Find all solutions when there are any.

Problem 7. Find ten other proofs of the Fundamental Theorem of Algebra. (No need to hand in)