Problem 1. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \rightarrow \mathbb{R}$. The function $f$ is said to be in $C^k(U)$, if $f$ has all continuous partial derivatives up to and including order $k$ on $U$. Let $l \leq k$ and $i_1, i_2, \ldots, i_l$ be a collection of integers between 1 and $l$. Under these conditions, show that for any permutation $\sigma \in S_l$, we have

$$D_{i_1, \ldots, i_l} f = D_{\sigma(i_1), \ldots, \sigma(i_l)} f.$$

Problem 2. Let

$$f(x, y) = \begin{cases} 0 & \text{when } (x, y) = (0, 0), \\ (x^2 + y^2) \sin \left( \frac{1}{\sqrt{x^2 + y^2}} \right) & \text{otherwise.} \end{cases}$$

Show that $D_1 f$ and $D_2 f$ exist everywhere but are not continuous at the origin.

Problem 3. Let $f : \mathbb{R}^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ be given by

$$f(x, y, z) = \left( \frac{x}{x^2 + y^2 + z^2}, \frac{y}{x^2 + y^2 + z^2}, \frac{z}{x^2 + y^2 + z^2} \right).$$

Show that $f$ is locally invertible at every point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Find an explicit formula for $f^{-1}$.

Problem 4. Consider the equations

$$ab^2 + cde + a^2d = 3 \quad \text{and} \quad ace^3 + 2bd - b^2e^2 = 2.$$ 

Determine for which pairs of variables the Implicit Function Theorem implies that the two equations can be uniquely solved for in terms of the other three near the point $(a, b, c, d, e) = (1, 1, 1, 1, 1)$.

Problem 5. Define a map from $\mathbb{R}^2$ to itself by setting

$$F(x, y) = (\sin x \cos y + \cos x \sin y, \cos x \cos y - \sin x \sin y).$$

Does there exist a point $(x_0, y_0)$ such that $F$ is locally invertible in a neighborhood of $F(x_0, y_0)$?