Problem 1. Show that if we add one more property to the definition of a $\sigma$-ring $\mathcal{M}$ on a set $X$ in 11.1 of the text: $\mathcal{M}$ contains the ambient set $X$, we will get a notion equivalent to the notion of a $\sigma$-algebra from class (a collection of subsets containing the empty set $\emptyset$ and closed under countable unions and complements).

Problem 2. Is the Cantor set a Borel set on $\mathbb{R}$?

Problem 3. Let $X$ be a separable metric space and let $\mathcal{M}$ be the $\sigma$-algebra generated by open balls in $X$. Show that $\mathcal{M}$ contains all the open sets in $X$ and all the closed sets. Describe some sets in an example $\mathcal{M}$ that are neither open nor closed. The $\sigma$-algebra $\mathcal{M}$ is called the $\sigma$-algebra of Borel sets in $X$. $\text{Hint}$: Show that every open set in $X$ is a countable union of open balls.

Problem 4. Can a $\sigma$-algebra in a set $X$ have cardinality $\aleph_0$? $\text{Hint}$: Show that if a $\sigma$-algebra is infinite, then it contains a countable collection of pairwise disjoint subsets. Deduce that the $\sigma$-algebra will also contain arbitrary unions of these disjoint subsets.

Problem 5. Show that if $\mu$ is a measure on a set $X$ with a $\sigma$-algebra $\mathcal{M}$ and there is a set $A \in \mathcal{M}$ such that $0 < \mu(A) < \infty$, then $\mu(\emptyset) = 0$.

Problem 6. Show that the $\sigma$-algebra generated by the collection of open rectangles (called intervals in the text) is the same as the $\sigma$-algebra generated by the collection of half-open rectangles.

Problem 7. Show that the Lebesgue outer measure of a face $I_1 \times \cdots \times I_{i-1} \times \{a\} \times I_{i+1} \times \cdots \times I_n$ of a rectangle $I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ is zero. Here $I_1, \ldots, I_n$ are open intervals in $\mathbb{R}$ and $a$ is one of the endpoints of $I_i$.

Problem 8. Show that the restriction $\mu^*|_E$ of an outer measure $\mu^*$ on a set $X$ to a subset $E \subset X$ defined by $\mu^*|_E(A) := \mu^*(E \cap A)$ for any $A \subset X$ is an outer measure on $X$. Show that any set that is measurable with respect to $\mu^*$ is measurable with respect to $\mu^*|_E$ (regardless of whether $E$ is measurable).