Math 5616H **Homework 9**

Posted: 4/6; Updated: 4/6; Due: Friday, 4/10/2015

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 11: Sections 8, 10-11.

Problem 1. Prove that any compact set K in $\mathbb{R}^n$ is Lebesgue measurable and $m(K) < \infty$.

Problem 2. Show that the Cantor set is measurable and has (Lebesgue) measure zero.

Problem 3. Let $c : [0, 1] \to \mathbb{R}$ be the Cantor function constructed in Homework 2. Extend c to all $\mathbb{R}$ by setting $c(x) := 0$ for $x \le 0$ and $c(x) := 1$ for $x \ge 1$. Let $f(x) := c(x) + x$. Show that f carries a set of (Lebesgue) measure zero (the Cantor set) to a set of positive measure. (This provides a counterexample, as one can show that f is a homeomorphism of $\mathbb{R}$, *i.e.*, a continuous map $\mathbb{R} \to \mathbb{R}$ which has a continuous inverse. You do not have to show that, though.)

Problem 4. A measure μ on a metric space with a given σ-algebra $\mathfrak{M}$ is called *regular*, if the σ-algebra of Borel sets is a subalgebra of $\mathfrak{M}$, and for every set A in $\mathfrak{M}$, we have

$$\mu(A) = \inf\{\mu(U) \mid U \text{ is open and } A \subset U\}$$

and

$$\mu(A) = \sup\{\mu(K) \mid K \text{ is compact and } K \subset A\}.$$

(1) Show that the counting measure on $\mathbb{Z}$ with the induced metric from $\mathbb{R}$ is regular. (2) Show that the delta measure with respect to a point x_0 on any metric space is regular.

Problem 5. Show that the Lebesgue measure on $\mathbb{R}^n$ is regular.