MATH 5616H: INTRODUCTION TO ANALYSIS II
SAMPLE MIDTERM EXAM II (WITH SOLUTIONS)

You may not use notes, books, etc. Only the exam paper, a pencil or pen may be kept on your desk during the test. Calculators are not allowed, either, but will not be needed. Ask me, and I will compute anything for you, if you need me to. Unless stated otherwise, please show all of your work and justify your answers in order to receive full credit.

Good luck!

Problem 1. Let B, C, and D respectively stand for *bounded*, *continuous*, and *differentiable*. For the three functions f, g, and $h : \mathbb{R}^2 \to \mathbb{R}$ that are 0 at $(0,0)$ and given by $\frac{xy}{x^2+y^2}$, $\frac{xy^2}{x^2+y^2}$, $\frac{x^2y^2}{x^2+y^2}$ elsewhere, say which functions have which properties and why.

Solution, suggested by Michael Hank. The simplest way to see that the last function $f(x,y)$ is differentiable at 0 is directly using the definition of the derivative $Df(0)$:

$$\lim_{(x,y)\to 0} \frac{f(x,y) - f(0) - Df(0)(x,y)}{(x^2+y^2)^{1/2}} = 0,$$

because

$$\begin{aligned}\frac{f(x,y) - f(0) - Df(0)(x,y)}{(x^2+y^2)^{1/2}} &= \frac{x^2y^2}{(x^2+y^2)^{3/2}} \\ &= |x| \cdot \frac{|x|}{\sqrt{x^2+y^2}} \cdot \frac{y^2}{x^2+y^2} \le |x|.\end{aligned}$$

□

Problem 2. Show that the system

$$\begin{cases} x - y \quad + z + u^2 = 2 \\ -x \quad + 2z + u^3 = 2 \\ \quad - y \ + 3z + u^4 = 3 \end{cases}$$

cannot be solved for x, y, and z in terms of u near the point $(x,y,z,u) = (1,1,1,1)$, but for any other group of three variables, a local C^1 solution in terms of the fourth variable is possible.

Date: April 24, 2015.

Solution. Let $F := x - y + z + u^2 - 2$, $G := -x + 2z + u^3 - 2$, $H := -y + 3z + u^4 - 3$. The function $(F, G, H) : \mathbb{R}^4 \to \mathbb{R}^3$ is of class C^1, its components being polynomials. At $(1, 1, 1, 1)$ we have $F = G = H = 0$ and

$$\det \frac{\partial(F,G,H)}{\partial(x,y,u)} \neq 0, \ \det \frac{\partial(F,G,H)}{\partial(y,z,u)} \neq 0, \ \det \frac{\partial(F,G,H)}{\partial(x,z,u)} \neq 0.$$

Thus, by the Implicit Function Theorem, we can solve the system above in a neighborhood of $(1, 1, 1, 1)$ for each group of three variables other than (x, y, z).

To see that we cannot solve the system for (x, y, z) near $(1, 1, 1, 1)$, we notice that the system is a linear system $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (2 - u^2, 2 - u^3, 3 - u^4)$ in $\mathbf{x} := (x, y, z)$ with A having determinant zero. Thus, as Calculus III teaches us, irrespective of the value of u, we will never have a unique solution of $A\mathbf{x} = \mathbf{b}$, if any. Solving in the context of Analysis means finding a unique solution. $\square$

Problem 3. Show that for any $\varepsilon > 0$ there exists an open set D dense in $\mathbb{R}^n$ such that the Lebesgue measure $m(D) < \varepsilon$.

Solution. Take the set Q of points with rational coordinates $\mathbb{R}^n$. This is a countable set. Enumerate it with the naturals. Surround the nth point of the set Q by an open rectangle of volume less than $\varepsilon/2^{n+1}$. Take for D the union of these rectangles. It is a Lebesgue measurable set, being open. Since the total volume of these rectangles will be less than ε, the Lebesgue outer measure of D will be less than ε. And for Lebesgue measurable sets, the Lebesgue measure is just the Lebesgue outer measure. (You can avoid considering the outer measure whatsoever by doing a simple exercise and proving that a measure is always countably subadditive on measurable sets.) $\square$

Problem 4. Suppose that s and t are nonnegative measurable simple functions on a measure space X with a measure μ. Show that

$$\int_X (s + t) d\mu = \int_X s d\mu + \int_X t d\mu.$$

Solution. Let $s = \sum_i a_i \chi_{A_i}$ and $t = \sum b_j \chi_{B_j}$ with pairwise disjoint A_i's and pairwise disjoint B_j's. (To come up with a solution plan, it is useful to look first at the case when s and t consist just of one characteristic function each.) Let $A := \bigcup_j A_j$ and $B := \bigcup_j B_j$. Fix one i. Then for each $x \in A_i$, we will have two options: $x \notin B$ or $x \in B_j$ for exactly one j. Take pairwise disjoint sets $A_i \setminus B$ and $A_i \cap B_j$ for all possible j.

Some of the latter sets may actually be empty. Note that

$$s + t = \sum_i a_i \chi_{A_i \setminus B} + \sum_{i,j} (a_i + b_j) \chi_{A_i \cap B_j} + \sum_j b_j \chi_{B_j \setminus A}$$

and

$$\int_X (s+t) d\mu = \sum_i a_i \mu(A_i \setminus B) + \sum_{i,j} (a_i + b_j) \mu(A_i \cap B_j) + \sum_j b_j \mu(B_j \setminus A)$$

$$= \sum_i a_i \left(\mu(A_i \setminus B) + \sum_j \mu(A_i \cap B_j) \right) + \sum_j b_j \left(\sum_i \mu(A_i \cap B_j) + \mu(B_j \setminus A) \right)$$

$$= \sum_i a_i \left(\mu(A_i \setminus B) + \mu(A_i \cap B) \right) + \sum_j b_j \left(\mu(A \cap B_j) + \mu(B_j \setminus A) \right)$$

$$= \sum_i a_i \mu(A_i) + \sum_j b_j \mu(B_j) = \int_X s d\mu + \int_X t d\mu.$$

To avoid organizing the complicated combinatorics above, one can use induction, first on the cardinality of the set of i's and then on the cardinality of the set of j's. This way one can just deal with two sets at a time. $\square$