MATH 5616H: INTRODUCTION TO ANALYSIS II
SAMPLE FINAL EXAM

You may not use notes, books, etc. Only the exam paper, a pencil or pen may be kept on your desk during the test. Calculators are not allowed, either, but will not be needed. Ask me, and I will compute anything for you, if you need me to. Unless stated otherwise, please show all of your work and justify your answers in order to receive full credit.

Good luck!

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $p > 0$, that is, $f(x+p) = f(x)$ for all x and p is the least positive number with this property. Prove that

$$\int_0^p f(x+y)dx = \int_0^p f(x)dx \qquad \text{for any } y \in \mathbb{R}.$$

Problem 2. (1) Find real numbers a and b such that the partial differential equation (you do not need to know what it is, just the equation below)

$$u_t(t,x) = (k-1)u_x(t,x) + u_{xx}(t,x) - ku(t,x), \qquad k \in \mathbb{R},$$

turns into $w_t(t,x) = w_{xx}(t,x)$ after the substitution

$$u(t,x) = e^{ax+bt}w(t,x).$$

(2) Find an equation relating the derivatives (a.k.a. differentials) $Du(t,x)$ and $Dw(t,x)$. If there are derivatives of other functions involved, compute them.

Problem 3. Let $\{f_i \mid i \in I\}$ be a uniformly bounded set of Riemann integrable functions on $[a,b] \subset \mathbb{R}$. Define

$$F_i(x) := \int_a^x f_i(t)dt, \qquad a \le x \le b.$$

Show that the family $\{F_i\}$ contains a uniformly convergent subsequence.

Problem 4. Suppose $f \in L^1([0,1])$, *i.e.*, $f : [0,1] \to [0,\infty]$ is an integrable function. Prove that $\lim_{\varepsilon \to 0+} \int_{[0,\varepsilon]} f dm = 0$.

Date: May 6, 2015.

Problem 5. Prove that if f is continuous on $[0,1]$, then
$$\lim_{n\to\infty}\int_0^1 f(x^n)dx = f(0).$$

Problem 6. Let X and Y be compact metric spaces and let $f(x,y) \in C(X\times Y)$ be a continuous real-valued function on $X\times Y$. Show that for every $\varepsilon > 0$ there exist $g_1,\ldots,g_n \in C(X)$ and $h_1,\ldots,h_n \in C(Y)$ such that
$$\left| f(x,y) - \sum_{i=1}^{n} g_i(x)h_i(y)\right| < \varepsilon \quad \text{for all } (x,y)\in X\times Y.$$

Problem 7. Show that any measure is countably subadditive.

Problem 8. Define $f:[0,1]\to\mathbb{R}$ by
$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,}\\ \frac{1}{\sqrt{d}} & \text{if } x \text{ is irrational and } x = 0.0\ldots 0d\ldots,\end{cases}$$
where d is the first nonzero digit in the decimal expansion of x. Prove that f is measurable.