This is Part I of the problem set.

I encourage you to cooperate with each other on the homeworks.

Reminder: all rings are commutative with an identity element 1, all ring homomorphisms carry 1 to 1, and a subring shares the same identity element with the ring.

**Problem 1.** Let \( A \) be a UFD. A polynomial
\[
a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \in A[X]
\]
is primitive if its coefficients \( a_i \) have no common factors in \( A \) (other than units).

Prove Gauss’ Lemma: the product of two primitive polynomials is primitive.

**Problem 2** (This problem is not for credit. You may do it for fun).

Prove the cases \( n = 3 \) and \( n = 4 \) of Fermat’s Last Theorem. [Harder, a hint will be given later. So far: use the discussion of \( \mathbb{Z}[\sqrt{1}] \) in the first lecture].

**Problem 3.** Let \( \phi : A \to B \) be a ring homomorphism. Prove that \( \phi^{-1} \) takes prime ideals of \( B \) to prime ideals of \( A \). [In particular, if \( A \subset B \) and \( P \) is a prime ideal of \( B \), then \( A \cap P \) is a prime ideal of \( A \)].

**Problem 4.** Prove or give a counterexample:

1. the intersection of two prime ideals is prime;
2. the ideal \( P_1 + P_2 \) generated by two prime ideals \( P_1, P_2 \) is again prime;
3. if \( \phi : A \to B \) is a ring homomorphism, then \( \phi^{-1} \) takes maximal ideals of \( B \) to maximal ideals of \( A \);
4. the map \( \phi^{-1} \) for a quotient homomorphism \( \phi : A \to A/I \) takes maximal ideals of \( A/I \) to maximal ideals of \( A \).

**Problem 5.**

1. If \( a \) is a unit and \( x \) is nilpotent, prove that \( a + x \) is again a unit.
2. Let \( A \) be a ring, and \( I \subset \text{nilrad} A \) an ideal; if \( x \in A \) maps to an invertible element of \( A/I \), prove that \( x \) is invertible in \( A \).

**Problem 6.** Show that if \( A \) is a reduced ring and has finitely many minimal prime ideals \( P_i \), i.e., minimal elements in the set of prime ideals of \( A \), then \( A \hookrightarrow \bigoplus_{i=1}^n A/P_i \); moreover the image has nonzero intersection with each summand.

**Problem 7.** Describe Spec \( \mathbb{R}[X] \) in terms of \( \mathbb{C} \).

**Problem 8.** Let \( A \) be a ring with zerodivisors, i.e., not an integral domain. Prove that \( A \) has either nonzero nilpotent elements, or more than one minimal prime ideal.

Date: September 10, 2003.