I encourage you to cooperate with each other on the homeworks.

Convention: all rings are commutative with an identity element 1 ≠ 0, all ring homomorphisms carry 1 to 1, and a subring shares the same identity element with the ring.

**Problem 1.** Find a ring $A$ and a multiplicative set $S$ such that the relation $(a, s) \sim (b, t) \iff at = bs$ is not an equivalence relation.

**Problem 2.** Let $M_i \subseteq M$ be submodules indexed by some set $J$, for which $M = \sum_{i\in J} M_i$, the sum of the submodules. Suppose that $S$ is a multiplicative set, and $S^{-1}M_i = 0$ for all $i \in J$. Make an original discovery concerning $S^{-1}M$.

**Problem 3.** If $S = \{1, f, f^2, \ldots \}$ is a multiplicative set of $A$, prove that $\text{Spec}(A_f) \subseteq \text{Spec} A$ is the complement of the closed set $\mathcal{V}(f)$.

**Problem 4.** Let $A = k[V] = k[X_1, \ldots, X_n]/I(V)$ be the coordinate ring of a variety $V \subset k^n$ and $f \in A$. Prove that $A[1/f]$ is the coordinate ring of a variety $V_f \subset k^{n+1}$, which is in natural bijective correspondence with the open set $V \setminus \mathcal{V}(f)$.

**Problem 5.** Exercise 2.2 of [E].

**Problem 6.** Exercise 2.8 of [E].

**Problem 7.** If $M$ is an $A$-module, show that $M$ can be identified with a certain subset of the sections of the surjection

$$\prod_{P\in \text{Spec} A} M_P \to \text{Spec} A,$$

$$m \mapsto P \quad \text{for } m \in M_P.$$  

If $S$ is a multiplicative set, show that $S^{-1}M$ can be identified with a subset of partially defined sections, defined for $P$ with $P \cap S = 0$. By the way, $M_P$ is called the stalk of $M$ over $P$.

**Problem 8.** Give an example of a ring $A$ and an ideal $I$ which is not primary, but satisfies the condition $fg \in I \implies f^n \in I$ or $g^n \in I$ for some $n$. [Hint: that is, find a nonprimary ideal whose radical is prime.]

**Problem 9** (Fitting’s Lemma). Let $M$ be a Noetherian module and $\phi : M \to M$ a homomorphism. Prove that $\ker\phi^n \cap \text{im}\phi^n = 0$ for some $n > 0$. [Hint: use our method of proving that every indecomposable ideal in a Noetherian ring is primary, when we were proving Noether’s theorem on the existence of primary decomposition.]

**Problem 10.** Let $A = k[X,Y,Z]$ and $I$ the ideal $(XY, X - YZ)$. Find a primary decomposition of $I$ and determine the corresponding primes. [Hint: to guess the result, draw the variety $V(I)$]. To prove it, note that the variety $X = YZ$ is
isomorphic to the $YZ$-plane; consider $A \rightarrow k[Y, Z]$ sending $X$ to $YZ$ and restriction of primary ideals via this homomorphism.]

**Problem 11.** Let $A = k[X, Y, Z]/(XZ - Y^2)$ and $P = (x, y)$, and set $M = A/P^2$.

1. Determine $\text{Ass } M$. [Hint: use a primary decomposition of $P^2$ in $A$.]
2. Find the elements of $M$ annihilated by each assassin.
3. Find a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ with $M_i/M_{i-1} \cong A/P_i$, where $P_i \in \text{Spec } A$, $i = 1, \ldots, n$.

**Problem 12.** Exercise 3.1 of [E].

**Problem 13.** Exercise 3.3 of [E].

**Problem 14.** Using Nakayama’s lemma, show that if $(A, m)$ is a Noetherian local ring, then the maximal ideal $m$ is principal, if and only if $m/m^2$ is one-dimensional over $k = A/m$. Show also that $A$ is a DVR, if and only if $A$ is Noetherian, local with $\text{Spec } A = \{0, m\}$, and $m/m^2$ is one-dimensional over $k = A/m$.

**Problem 15 (DVRs and nonsingular curves).** This problem says that $A$ is a DVR, if and only if the plane curve $C := \{f = 0\} \subset k^2$ is nonsingular at $(0, 0)$.

Let $k$ be an algebraically closed field and $f \in k[X, Y]$ an irreducible nonconstant polynomial of the form

$$f(X, Y) = l(X, Y) + g(X, Y)$$

with $l(X, Y) = aX + bY$ and $g \in (X, Y)^2$. Set $R = k[X, Y]/(f)$, $P = (X, Y)/(f)$, and $(A, m) = (R_P, m_P)$. Prove that $A$ is a DVR, if and only if $l \neq 0$. [Hint: use Problem 14.]

That is it! Happy Thanksgiving!