Math 8212 **Homework 6**
Posted: 4/7/2020; Chapter 19 removed from reading: 4/25; Due: Monday, 4/20/2020

The problem set is due at the beginning of the class on Monday, April 20.

All homework is assigned based on Eisenbud's third printing (1999 or later). If you are using an older printing, make sure to check the correction lists on our class home page against the assigned exercises.

Reading: **Class notes. Eisenbud**: Sections 15.3–15.4, 17.1–17.3. **Cox, Little, O'Shea**: Sections 2.5–2.7.

Problem 1. Let $I = (xz - y^2, yw - z^2, yz - xw) \lhd \mathbb{Q}[x, y, z, w]$ and let $f = x^2y^2w^2 - y^4z^2$. Use the division algorithm (by hand!) to determine whether or not f lies in I.

Problem 2. Find an example of an ideal I which is generated by quadrics but where $\mathrm{in}(I)$ has a minimal generator of degree at least 4. You can choose any monomial order $>$ you like.

Problem 3. Let $>$ be a monomial ordering on $S := k[x_1, \ldots, x_n]$ that respects degree, *i.e.*, $x^\alpha > x^\beta$ whenever $|\alpha| > |\beta|$. For an arbitrary (*i.e.*, not necessarily homogeneous) polynomial $g \in S$ of degree d, we define the *homogenization of g by x_0* as

$$g^h(x_0, x_1, \ldots, x_n) := x_0^d \cdot g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}).$$

For an ideal $I \lhd S$, define

$$I^h := (g^h \mid g \in I) \lhd k[x_0, \ldots, x_n].$$

Let $\mathcal{G} = \{g_1, \ldots, g_r\}$ be a Gröbner basis of I. Prove that $I^h = (g_1^h, \ldots, g_r^h)$.

Problem 4. Using the lex monomial order, compute a reduced Gröbner basis (by hand!) for the ideal

$$(xy - x - 2y + 2, x^2 + xy - 2x) \lhd \mathbb{C}[x, y].$$

Use this to determine all solutions in $\mathbb{C}^2$ to the system of equations

$$xy - x - 2y + 2 = 0,$$
$$x^2 + xy - 2x = 0.$$

Bonus: Instead of $\mathbb{C}$, we could have worked over a field k of positive characteristic. Over which characteristics would the Gröbner basis have been the same?

Problem 5. Given two complexes $F_\bullet$ and $G_\bullet$, we can define a new complex by

$$(F \otimes G)_i := \sum_{p+q=i} F_p \otimes G_q$$

with boundary

$$\partial(f \otimes g) := \partial f \otimes g + (-1)^p f \otimes \partial g.$$

Fix a (commutative, Noetherian) ring R and $x_1, x_2, x_3 \in R$. Confirm that, with this definition, there exists a natural isomorphism (up to sign):

$$\mathbf{K}(x_1, x_2) \otimes \mathbf{K}(x_3) \cong \mathbf{K}(x_1, x_2, x_3).$$

Problem 6. Eisenbud's Exercise 17.1.