Math 8253 **Homework 1**
Posted: 09/12/2017; 09/20; Hint added: 09/26; Due: Wednesday, 09/27

Reading: Syllabus. **Class notes**. **Vakil**: Sections 1.2, 1.3.10-11, 3.1, 3.2 before 3.2.9, 3.3, 3.4, 3.5, 3.6, 3.7. **Hartshorne**: Pages 1-2 before the definition of an algebraic set; *cf.* Proposition 3.5 and its corollaries in Hartshorne's Chapter I. p. 58-59; Section II.2 before graded rings, ignoring morphisms and sheaves for the time being.

Problem 1. Prove that two categories C and D are equivalent, if and only if there exists a functor $F: C \to D$ which is bijective on morphisms ($\operatorname{Mor}_C(c_1, c_2) \xrightarrow{F} \operatorname{Mor}_D(F(c_1), F(c_2))$ is a bijection for any $c_1, c_2 \in \operatorname{Obj} C$) and for every object d of D, and isomorpshim $d \xrightarrow{\sim} F(c)$ is given for some object c of C. Assume the following, standard definition of *equivalence of categories*: there exist functors $F: C \to D$ and $G: D \to C$ such that the compositions FG and GF are naturally equivalent to the identity functors.

Problem 2. Let A_1 and A_2 be two rings. Their Cartesian product $A_1 \times A_2$ becomes a ring under componentwise addition and multiplication. Find a bijection $\operatorname{Spec} A_1 \times A_2 \xrightarrow{\sim} \operatorname{Spec} A_1 \coprod \operatorname{Spec} A_2$.

Problem 3. Show that an open subset $U \subset \operatorname{Spec} A$ containing all closed points of $\operatorname{Spec} A$ must coincide with $\operatorname{Spec} A$. *Hint*: Show first that every nonempty closed subset $V \subset \operatorname{Spec} A$ contains a closed point.

Problem 4. Let $k[t]$ be the polynomial ring in one variable over an algebraically closed field k. Show that the set of closed points in $\operatorname{Spec} k[t]$ can be identified with k and that there is precisely one nonclosed point, namely the generic point.

Problem 5. Now consider the polynomial ring $k[t_1, t_2]$ in two variables, still over an algebraically closed field k. Let $X = \operatorname{Spec} k[t_1, t_2]$. Prove the following statements.

(1) The set of closed points in X may be identified with k^2.
(2) The nonclosed points of X other than the generic point are given by the ideals of type $(f) \subset k[t_1, t_2]$, where $f \in k[t_1, t_2]$ is irreducible.
(3) The closure $\overline{\{x\}}$ of a point $x \in X$ as in (2) consists of x itself as a generic point and the "curve" $\{x \in k^2 \mid f(x) = 0\}$.

Problem 6. Let U be a nonempty open set in an irreducible topological space X. Show that U is also irreducible.

Problem 7. Let k be a finite field and A be a *finite k-algebra*, *i.e.*, a k-algebra such that $\dim_k A < \infty$. Provide the *maximal spectrum* $\operatorname{Spm} A$, *i.e.*, the subset of $\operatorname{Spec} A$ consisting of maximal ideals, with the induced topology. Show that $\operatorname{Spm} A$ is Hausdorff. *Challenge question* (outside of the homework): Is the same true for $\operatorname{Spec} A$?

Problem 8. Let A be an *algebra of finite type* over a field k. (This means A is finitely generated as an algebra or that there exists a surjective algebra homomorphism $k[t_1, \dots, k_n] \to A$.) Consider a closed subset $Y \subset \operatorname{Spec} A$. Show that the closed points are dense in Y. *Hint*: Use the fact, which you do not have to prove, stating that the *Jacobson radical* $J(A) := \bigcap_{\mathfrak{m} \in \operatorname{Spm} A} \mathfrak{m}$ of A coincides with its nil-radical $\operatorname{rad}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$ for any algebra A of finite type. Almost any text on Commutative Algebra (excluding Atiyah-Macdonald) has this statement. It also implies $I(V(\mathfrak{a}) \cap \operatorname{Spm} A) = \operatorname{rad}(\mathfrak{a})$, which is a more traditional form of Hilbert's Nullstellensatz than the simpler version $I(V(\mathfrak{a})) = \operatorname{rad}(\mathfrak{a})$ we had in class.