Math 8253 **Homework 4**

Posted: 11/05/2017; Due date extended on 11/13; Due: Wednesday, 11/15

The problem set is due at the beginning of the class on Wednesday, November 15.

Reading: **Class notes**. **Vakil**: Sections 2.5, 3.2.3-I, 4.1.D-4, 4.3, 6.3 through 6.3.D, 13.2-3. **Hartshorne**: Sections II.1 (pp. 63–64), II.2 before graded rings and Proj, II.5 through 5.7, skipping direct and inverse images f_* and f^*.

Problem 1. Describe all open sets of the affine scheme $X = \operatorname{Spec} \mathbb{C}[t]/(t^2 - t)$ and the restriction morphisms of its structure sheaf $\mathcal{O}_X$.

Problem 2. Show that $\operatorname{Spec} \mathbb{Z}$ is a *terminal object* in the category of schemes, that is to say, for any scheme X, there exists a unique scheme morphism $X \to \operatorname{Spec} \mathbb{Z}$. Show that this implies that the category of schemes is equivalent to the category of schemes over $\operatorname{Spec} \mathbb{Z}$.

Problem 3. Consider the *cuspidal cubic* $X = \operatorname{Spec} K[x,y]/(y^2 - x^3)$ over a field K. Consider the homomorphism

$$\begin{aligned} \varphi : K[x,y]/(y^2 - x^3) &\to K[t], \\ x &\mapsto t^2, \\ y &\mapsto t^3, \end{aligned}$$

of K-algebras. Show that the induced morphism $\mathbb{A}^1_K \to X$ of schemes is not an isomorphism, even though it is a homeomorphism of the underlying topological spaces. *Hint*: Note that φ is not onto. Show that the localization of φ by the multiplicative system generated by x is an isomorphism of K-algebras.

Problem 4. Suppose $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups on a topological space X. For any open set $U \subset X$, set

$$\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})(U) := \operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where $\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is the set of sheaf morphisms on U. Define the structure of a presheaf of abelian groups on $\underline{\operatorname{Hom}}$ and show it is actually a sheaf.

Problem 5. Given a **closed** point x in a topological space X and an abelian group F, define a presheaf $\mathcal{F}$ on X by

$$\mathcal{F}(U) := \begin{cases} F & \text{if } x \in U, \\ 0 & \text{otherwise,} \end{cases}$$

for each open $U \subset X$.

(1) Show that this presheaf $\mathcal{F}$ is a sheaf. It is called the *skyscraper sheaf* on X with stalk F at x.
(2) Show that this sheaf is uniquely characterized up to canonical isomorphism by the fact that $\mathcal{F}_x = F$ and $\mathcal{F}_y = 0$ for $y \neq x$. (*Canonical* is a loose term, here meaning "given by a construction independent of the particulars of X and F". You do not have to show this independence of your construction.)

Problem 6. Exercise 13.2.A from Vakil (June 4, 2017 version).

Problem 7. Give a more detailed proof of Hartshorne's Proposition II.5.2(c).